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AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

An Analysis of the Behavior of a Multi-Grid Spherical Sensor in a Drifting Maxwellian Plasma

MICHAEL SMIDDY
ROBERT D. STUART

OFFICE OF AEROSPACE RESEARCH
United States Air Force

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IONOSPHERIC PHYSICS LABORATORY PROJECT 8617

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Abstract

A theoretical derivation of the characteristic of a spherical probe in a flowing Maxwellian plasma is given together with various limiting forms of the equations. The limitations imposed on the validity of the theory by the characteristics of the sheath are discussed. The theory is then extended to the case of an n-grid sensor. Finally, the application of the theory to the determination of plasma parameters from the probe characteristic is discussed.

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An Analysis of the Behavior of a Multi-Grid Spherical Sensor in a Drifting Maxwellian Plasma

1. INTRODUCTION

The use of a probe to determine the parameters of a plasma has long been one of the standard methods of experimental measurement. The foundations of probe theory were laid down in a classic series of papers by Langmuir.

In this paper, the details of the calculation are given for the current flow to a sphere in a plasma containing charged particles, where these particles have a Maxwellian velocity distribution with a superimposed drift. The current is obtained as a function of the potential difference between the sphere and the plasma and, following Langmuir, the retarding and accelerating cases are treated separately. The discussion in Section 7 imposes an important limitation on the conditions under which the expressions can be considered valid. The general theory of an n-grid sensor is developed in Section 10, and the particular case of a two-element sensor is discussed in Section 11. This last section, using various special cases of the expressions developed earlier, discusses the problem of obtaining the parameters describing the plasma; namely—density, temperature, mass, and plasma potential, from an observed current-voltage curve. A summary of the equations with the numerical constants most useful in practical applications is given in the appendix.

(Received for publication 11 December 1968)

2. THE INTEGRAL FOR THE CURRENT

The current crossing an element of area $\underline{dS}$ is given by $\underline{J} \cdot \underline{dS}$ where $\underline{J}$ is the current density. If $\underline{J}$ is due to the motion of charged particles then we can write $\underline{J} = Ne\underline{v}$, where N is the charged particle density, e the charge on each of the particles, and $\underline{v}$ their velocity. If the charged particles have a velocity distribution, then the number of particles with velocities in the range $\underline{v}$, $\underline{v} + \underline{dv}$ may be written $N\, f(\underline{v})\, dv$, and the current density due to this group of particles is $Ne\, f(\underline{v})\, dv$. The current flowing across the element of area $\underline{dS}$ is $Nef(\underline{v})\, dv\, (\underline{v} \cdot \underline{dS})$. The total current flowing across a surface S due to all of the charged particles can be written

$$I = \int_S \int_{\underline{v}} Ne\, f(\underline{v})\, dv\, (\underline{v} \cdot \underline{dS}) \quad ,$$

where the surface integral is taken over the surface under consideration, and the velocity integral, over the range of velocities possessed by particles which cross that surface.

We consider a spherical collector of radius R situated in a gas in which the velocity distribution of the charged particles is Maxwellian and there is, in addition, a superimposed steady drift motion relative to the sphere. To calculate the current to the sphere, we surround it by a concentric spherical surface of radius r and calculate the current flowing between the two spheres. The problem now consists of developing the appropriate distribution function for a Maxwell distribution with a superimposed drift, calculating the appropriate limits for the velocity integral in terms of the potential difference between the two surfaces, and finally evaluating the integral.

3. The distribution function

The Maxwell distribution function may be written

$$f(v'_x, v'_y, v'_z) = \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp\left[-a\left(v'^2_x + v'^2_y + v'^2_z\right)\right] \quad ,$$

where v'_x, v'_y, v'_z, are the components of the velocity parallel to a rectangular set of axes, x'y'z', at rest in the gas; and $a = m/2kT$ where m is the mass of the particle, T is the temperature of the gas, and k is Boltzmann's constant. If we now superimpose a steady drift v_0 in the z direction, then the velocities v_x, v_y, v_z, in a set of axes xyz at rest in the sphere, become $v_x = v'_x$, $v_y = v'_y$, and $v_z = v'_z + v_0$.

To calculate the current crossing a spherical surface of radius r we require $\underline{v} \cdot \underline{dS}$ which can be written $v_r r^2 \sin\theta\, d\theta\, d\phi$, where v_r is the radial component of the

velocity, and θ and ϕ are the colatitude and longitude angles in conventional spherical coordinate notation. We must, therefore, express the distribution function in terms of the radial velocity. The velocity of the particles can be represented in terms of v_r, v_θ, v_ϕ, which are the components in a local cartesian system representing the rectangular components of velocity in the directions of increasing r, θ, and ϕ at the particular point concerned. Since v_x, v_y, v_z and v_r, v_θ, v_ϕ are two sets of rectangular components relative to axes which are at rest in the sphere, then

$$v_x^2 + v_y^2 + v_z^2 = v_r^2 + v_\theta^2 + v_\phi^2$$

and

$$\begin{aligned} v'^2_x + v'^2_y + v'^2_z &= v_x^2 + v_y^2 + (v_z + v_0)^2 \\ &= v_x^2 + v_y^2 + v_z^2 + v_0^2 + 2v_0v_z \\ &= v_r^2 + v_\theta^2 + v_\phi^2 + v_0^2 + 2v_0(v_r \cos\theta - v_\theta \sin\theta) \quad . \end{aligned}$$

In the transformed system, the volume element $dv_x dv_y dv_z$ becomes $dv_r dv_\theta dv_\phi$ since this is still a rectangular system. To evaluate the integral, a further transformation is required into a spherical coordinate system in velocity space. The colatitude ϵ is measured from the v_r direction and the longitude λ is the plane defined by v_θ and v_ϕ from the v_θ direction.

Carrying through the appropriate substitutions gives

$$v'^2_x + v'^2_y + v'^2_z = v^2 + v_0^2 + 2vv_0(\cos\epsilon \cos\theta - \sin\epsilon \cos\lambda \sin\theta)$$

where v is the total velocity of the particle.

The volume element now becomes $v^2 \sin\epsilon \, dv \, d\epsilon \, d\lambda$ and the integral for the current crossing the sphere is

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{v} \int_{\epsilon} \int_{\lambda} Ne\left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp\left[-a\left\{v^2 + v_0^2 + 2vv_0(\cos\epsilon \cos\theta - \right.\right.$$
$$\left.\left. - \sin\lambda \cos\lambda \sin\theta)\right\}\right](v\cos\epsilon)(v^2 \sin\epsilon)\, d\lambda\, d\epsilon\, dv\, (r^2 \sin\theta)\, d\theta\, d\phi .$$

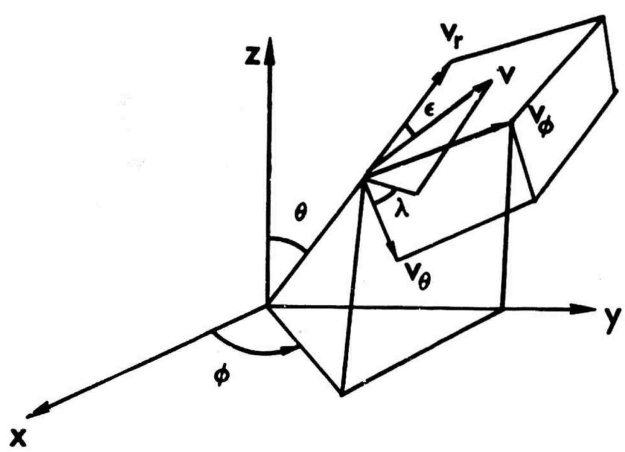

Figure 1. The Coordinate System

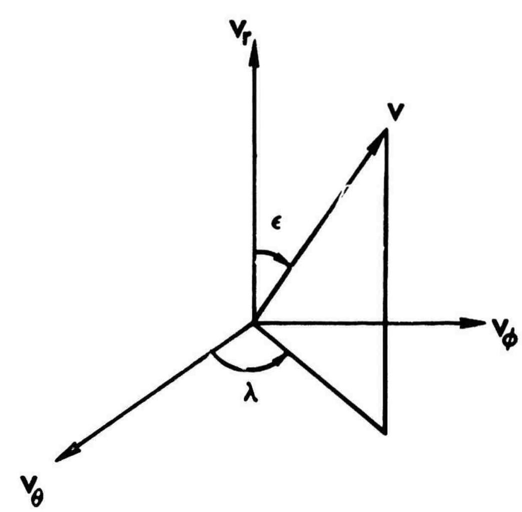

Figure 2. The Coordinate System

The convention followed so far is that the positive direction is outward from the center of the sphere, so that the current I given by this expression is the current flowing outward. Since we are interested in the current flowing to a spherical electrode in the plasma we need the current flowing inwards and it is convenient to change from the convention at this point. The velocity v is always positive but we now measure its colatitude from the inward directed radius vector, rather than the outward. Hence, if we write $\epsilon' = \pi - \epsilon$, then ϵ' varies over the range 0 to $\pi/2$ for inward flowing currents. The expression for I now becomes

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{v} \int_{\epsilon'} \int_{\lambda} Ne\left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp\left[-a\left\{v^2 + v_0^2 - 2vv_0 (\cos\epsilon' \cos\phi + \right.\right.$$

$$\left.\left. + \sin\epsilon' \cos\lambda \sin\theta)\right\}\right] v^3 (-\cos\epsilon') \sin\epsilon' \, d\lambda \, (-d\epsilon') \, dv \, r^2 \sin\theta \, d\theta \, d\phi \quad .$$

We shall, henceforth, drop the prime from ϵ, it now being understood that ϵ is measured from the inward directed radius vector.

The integrand is not a function of ϕ, which we would expect from the axial symmetry of the problem about the drift velocity. Hence, we can write

$$I = 2\pi r^2 Ne\left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp(-av_0^2) \int_{\theta=0}^{\pi} \int_{v} \int_{\epsilon} \int_{\lambda} v^3 \exp\left[-a\left\{v^2 - 2vv_0\right.\right.$$

$$\left.\left.(\cos\epsilon \cos\theta + \sin\lambda \cos\lambda \sin\theta)\right\}\right] \cos\epsilon \sin\epsilon \sin\theta \, d\lambda \, d\epsilon \, dv \, d\theta \quad .$$

Note that the symmetry of the problem also requires that the same current be obtained regardless of the sign of v_0. (This is not immediately obvious from the above expression.)

We now need the limits for the velocity integral.

1. LIMITS OF INTEGRATION

We assume that the velocity distribution of the particles at radius r is given by the distribution function derived above. We also assume a potential difference V between this surface and the spherical collector radius R. We now calculate the

possible range of velocities a particle can have at radius r if it is to be able to reach the collector radius R.

The calculation divides into two cases: the retarding case when the sign of the charge on the particle and the sign of the potential on the sphere are the same, and the accelerating case when they are opposite. These two cases were distinguished by Langmuir.

Considering first the retarding case, then a particle with velocity v travelling at an angle ϵ to the (inward)radius vector at radius r will arrive at the collector radius R with velocity v_1 travelling at an angle ϵ_1 to the radius vector where these quantities are related by the laws of conservation of angular momentum and energy. If the potential difference between the collector and the surface radius r is V, then from energy considerations

$$\frac{1}{2} mv^2 = \frac{1}{2} mv_1^2 + |eV| \quad .$$

We define a velocity v_R by the relation $\frac{1}{2} mv_R^2 = |eV|$ and this equation then becomes

$$v^2 = v_1^2 + v_R^2 \quad .$$

From angular momentum considerations

$$rv \sin\epsilon = Rv_1 \sin\epsilon_1 \quad .$$

The particle will be collected if it arrives at the collector travelling tangentially ($\sin {}_1 \leq 1$), that is, if

$$\frac{rv \sin\epsilon}{R} \leq v_1 \quad ,$$

which on substituting in the energy relation gives

$$v^2 - v_R^2 = v_1^2 \geq \frac{r^2 v^2 \sin^2\epsilon}{R^2} \quad .$$

Hence, for a given value of v, the particle is collected if

$$\sin^2\epsilon \leq \frac{R^2}{r^2}\left(1 - \frac{v_R^2}{v^2}\right) ,$$

with the limiting value, $\sin \epsilon_R$, being given by the equality sign. The limits for the ϵ integration are, therefore, 0 and ϵ_R.

Note that the upper limit of ϵ is a function of v, one of the other variables in the integration. We obtain the limits for v by noting that the smallest value of $\sin^2\epsilon$ is zero and, hence, $v^2 - v_R^2 \geq 0$ or $v \geq v_R$. The limits on v, therefore, become v_R and ∞. Finally, for the limits on λ we note that the arguments given above are valid for all values of λ, and, hence, the limits for the λ integration are 0 and 2π.

The expression for the current now becomes

$$I = 2\pi r^2 \, Ne\left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp(-av_0^2) \int_{\theta=0}^{\pi} \int_{v_R}^{\infty} \int_{0}^{\epsilon_R(v)} \int_{\lambda=0}^{2\pi} v^3$$

$$\exp\left[-a\left\{v^2 - 2vv_0(\cos\epsilon\cos\theta + \sin\epsilon\cos\lambda\sin\theta)\right\}\right] \cos\epsilon \sin\epsilon \sin\theta \, d\lambda \, d\epsilon \, dv \, d\theta \ .$$

For the accelerating case, e and V now have opposite signs and the energy relation can be written

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 - |eV| \ .$$

Defining a velocity v'_A by the relation $\frac{1}{2}mv'^2_A = |eV|$, we have

$$v^2 = v_1^2 + v'^2_A \ .$$

Following the same steps as before,

$$v^2 + v'^2_A \geq \frac{r^2 v^2 \sin^2\epsilon}{R^2} \ .$$

The integral must now be evaluated in two parts. The above relation is satisfied for all values of ϵ from 0 to $\pi/2$ if

$$v^2 + v'^2_A \geq \frac{r^2 v^2}{R^2}$$

or

$$v^2 \leq v'^2_A \frac{R^2}{r^2 - R^2} \quad .$$

Defining $\alpha = R^2/(r^2 - R^2)$ and $V_A^2 = \alpha\, v'^2_A$, the limits for the first part of the integral become 0 to $\pi/2$ for ϵ and 0 to v_A for v. (Note $\frac{1}{2} m v_A^2 = \alpha\ |eV|$.) The current I may now be written

$$I_1 = 2\pi r^2 N e \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp(-a v_0^2) \int_{\theta=0}^{\pi} \int_0^{v_A} \int_{\epsilon=0}^{\frac{\pi}{2}} \int_{\lambda=0}^{2\pi} v^3$$

$$\exp\left[-a\left\{v^2 - 2vv_0(\cos\epsilon\,\cos\theta + \sin\epsilon\,\cos\lambda\,\sin\theta)\right\}\right] \cos\epsilon\,\sin\epsilon\,\sin\theta\, d\lambda\, d\epsilon\, dv\, d\theta \quad .$$

If the particle has a velocity greater than v_A, it will be collected if

$$\sin^2\epsilon \leq \frac{R^2}{r^2}\left(1 + \frac{v'^2_A}{v^2}\right) \quad ,$$

or if

$$\sin^2\epsilon \leq \frac{\alpha}{\alpha + 1}\left(1 + \frac{v_A}{\alpha v^2}\right) .$$

Again, the limiting value ϵ_A is given by the equality sign. Hence, the second part of the integral becomes

$$I_2 = 2\pi r^2 N e\left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp(-av_0^2) \int_{\theta=0}^{\pi} \int_{v_A}^{\infty} \int_{0}^{\epsilon_A(v)} \int_{\lambda=0}^{2\pi} v^3$$

$$\exp\left[-a\left\{v^2 - 2vv_0\,(\cos\epsilon\,\cos\theta + \sin\epsilon\,\cos\lambda\,\sin\theta)\right\}\right] \cos\epsilon\,\sin\epsilon\,\sin\theta\;d\lambda\,d\epsilon\,dv\,d\theta\;.$$

5. EVALUATION OF THE INTEGRAL

The three integrals formulated in the last section are identical as far as the θ and λ integrations are concerned, so we evaluate these first. The integration required is

$$\int_{\theta=0}^{\pi} \int_{\lambda=0}^{2\pi} \exp\left[2a\,vv_0(\cos\epsilon\,\cos\theta + \sin\epsilon\,\cos\lambda\,\sin\theta)\right] \sin\theta\;d\lambda\,d\theta\;.$$

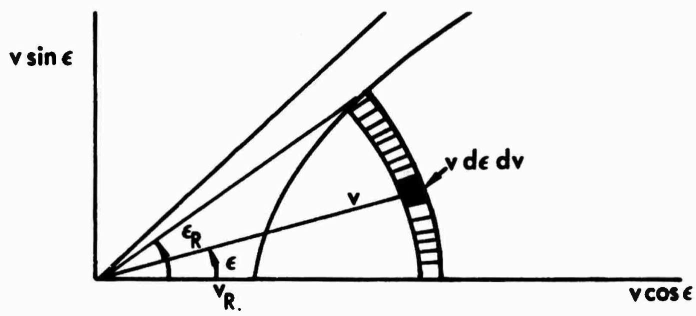

Figure 3. Limits of Integration

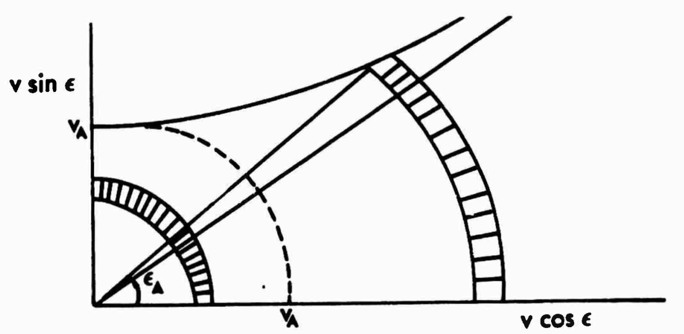

Figure 4. Limits of Integration

We can regard this as an integration over the unit sphere in $\theta\lambda$ space. The direction cosines of a point on the unit sphere are given by $\ell = \sin\theta \cos\lambda$, $m = \sin\theta \sin\lambda$, $n = \cos\theta$. Writing $A = 2a\,vv_0$, we have for the integral

$$\iint_S \exp A\,(n \cos\epsilon + \ell \sin\epsilon)\, dS,$$

where S represents the unit sphere and dS is the element of area $\sin\theta\, d\lambda\, d\theta$ in spherical polar coordinates. Making a cyclical notation of the direction cosine gives

$$\iint_S \exp A\,(n \cos\epsilon + \ell \sin\epsilon)\, dS = \iint_S \exp A\,(\ell \cos\epsilon + m \sin\epsilon)\, dS =$$

$$= \int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} \exp\left\{A(\sin\theta \cos\lambda \cos\epsilon + \sin\theta \sin\lambda \sin\epsilon)\right\} \sin\theta\, d\theta\, d\lambda =$$

$$= \int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} \exp\left\{A \sin\theta \cos(\lambda - \epsilon)\right\} \sin\theta\, d\theta\, d\lambda \quad .$$

Substituting $\lambda' = \lambda - \epsilon$ gives

$$\int_{\theta=0}^{\pi}\int_{\lambda'=-\epsilon}^{2\pi-\epsilon} \exp\left\{A \sin\theta \cos\lambda'\right\} \sin\theta\, d\theta\, d\lambda' \quad ,$$

and, since the integrand is periodic in λ' and we are integrating over a complete period, any complete period can be used. Therefore,

$$\int_{\theta=0}^{\pi}\int_{\lambda'=-\epsilon}^{2\pi-\epsilon} \exp\left\{A \sin\theta \cos\lambda'\right\} \sin\theta\, d\theta\, d\lambda' = \int_{\theta=0}^{\pi}\int_{\lambda'=0}^{2\pi} \exp\left\{A \sin\theta \cos\lambda'\right\}$$

$$\sin\theta\, d\theta\, d\lambda' = \int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} \exp\left\{A \sin\theta \cos\lambda\right\} \sin\theta\, d\theta\, d\lambda = \iint_S \exp(A\ell)\, dS =$$

$$= \iint_S \exp(An)\, dS = \int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} \exp(A\cos\theta)\sin\theta\, d\theta\, d\lambda =$$

$$= \left[-\frac{2\pi}{A}\exp(A\cos\theta)\right]_0^{\pi} = -\frac{\pi}{avv_0}\left\{\exp(-2avv_0) - \exp(2avv_0)\right\} \quad .$$

Substituting the above result into the expression for current gives

$$I = 2\pi r^2 Ne\left(\frac{a}{\pi}\right)^{\frac{3}{2}} \exp(-av_0^2)\int_v\int_\epsilon \exp(-av^2)\frac{(-\pi)}{avv_0}\left\{\exp(-2avv_0) - \exp(2avv_0)\right\}$$

$$\cos\epsilon\,\sin\epsilon\, d\epsilon\, dv \quad .$$

and so

$$I = 2\pi r^2 Ne\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \frac{1}{v_0}\int_v\int_\epsilon v^2\left[\exp\left\{-a(v-v_0)^2\right\} - \exp\left\{-a(v+v_0)^2\right\}\right]$$

$$\cos\epsilon\,\sin\epsilon\, d\epsilon\, dv \quad .$$

We now continue this integration for the retarding case. For the ϵ integral we have

$$\int_0^{\epsilon_R} \cos\epsilon\,\sin\epsilon\, d\epsilon = \frac{1}{2}\sin^2\epsilon_R = \frac{1}{2}\,\frac{R^2}{r^2}\left(1 - \frac{v_R^2}{v^2}\right) \quad .$$

Substituting this result gives

$$I = \pi R^2 Ne\sqrt{\frac{a}{\pi}}\,\frac{1}{v_0}\int_{v_R}^{\infty}(v^2 - v_R^2)\left[\exp\left\{-a(v-v_0)^2\right\} - \exp\left\{-a(v+\gamma_0)^2\right\}\right] dv \quad .$$

These integrals may be evaluated in terms of the error function which we define as as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\, dt \quad .$$

Using this definition, we have

$$\int_{v_R}^{\infty} v^2 \exp\left\{-a(v - v_0)^2\right\} dv = \left(\frac{1}{2a} + v_0^2\right) \frac{1}{2} \sqrt{\frac{\pi}{a}} \left\{\mathrm{erf}\, a^{\frac{1}{2}} (v_R - v_0)\right\} + \frac{1}{2a} (v_R + v_0)$$

$$\exp\left\{-a(v_R - v_0)^2\right\} ,$$

$$\int_{v_R}^{\infty} v^2 \left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a(v + v_0)^2\right\}\right] dv = \frac{1}{2} \sqrt{\frac{\pi}{a}} \left(\frac{1}{2a} + v_0^2\right)$$

$$\left[\mathrm{erf}\, a^{\frac{1}{2}} (v_R + v_0) - \mathrm{erf}\, a^{\frac{1}{2}} (v_R - v_0)\right] + \frac{v_R + v_0}{2a} \exp\left[-a(v_R - v_0)^2\right] -$$

$$- \frac{v_R - v_0}{2a} \left[\exp -a (v_R + v_0)^2\right]$$

$$\int_{v_R}^{\infty} v_R^2 \exp\left\{-a(v - v_0)^2\right\} dv = \frac{1}{2} \sqrt{\frac{\pi}{a}}\, v_R^2 \left[1 - \mathrm{erf}\, a^{\frac{1}{2}} (v_R - v_0)\right]$$

$$\int_{v_R}^{\infty} v_R^2 \left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a(v + v_0)^2\right\}\right] dv =$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{a}}\, v_R^2 \left[\mathrm{erf}\, a^{\frac{1}{2}} (v_R + v_0) - \mathrm{erf}\, a^{\frac{1}{2}} (v_R - v_0)\right] .$$

On substituting in these results and simplifying, we get

$$I = \pi R^2 Ne \sqrt{\frac{a}{\pi}} \frac{1}{v_0} \left[\frac{1}{2} \sqrt{\frac{\pi}{a}} \left(\frac{1}{2a} + v_0^2 - v_R^2\right) \left\{\mathrm{erf}\, a^{\frac{1}{2}} (v_R + v_0) - \mathrm{erf}\, a^{\frac{1}{2}} (v_R - v_0)\right\} + \right.$$

$$\left. + \frac{v_R + v_0}{2a} \exp\left\{-a(v_R - v_0)^2\right\} - \frac{v_R - v_0}{2a} \exp\left\{-a(v_R + v_0)^2\right\}\right] .$$

It is convenient to define two new parameters

$$\gamma^2 = av_0^2 = \frac{m}{2kT} v_0^2 \quad , \qquad x^2 = av_R^2 = \frac{|ev|}{kT} \quad .$$

Finally, we have

$$I = \pi R^2 \, Ne \frac{1}{\sqrt{\pi a}} \, \frac{1}{\gamma} \, \frac{\sqrt{\pi}}{2} \left(\frac{1}{2} + \gamma^2 - x^2 \right) \left\{ \operatorname{erf}(x + \gamma) - \operatorname{erf}(x - \gamma) \right\} +$$

$$+ \frac{1}{2}(x + \gamma) \exp\left\{ -(x - \gamma)^2 \right\} - \frac{1}{2}(x - \gamma) \exp\left\{ -(x + \gamma)^2 \right\} \Big] \quad .$$

Turning now to the accelerating form of the expression, the ϵ integral for the first part is simply

$$\int_0^{\pi/2} \cos\epsilon \, \sin\epsilon \, d\epsilon = \frac{1}{2} \quad ,$$

giving for I_1

$$I_1 = \pi r^2 \, Ne \sqrt{\frac{a}{\pi}} \, \frac{1}{v_0} \int_0^{v_A} v^2 \left[\exp\left\{ -a(v - v_0)^2 \right\} - \exp\left\{ -a(v + v_0)^2 \right\} \right] dv \quad .$$

This can again be evaluated in terms of the error function

$$\int_0^{v_A} v^2 \exp\left\{ -a(v - v_0)^2 \right\} dv = \frac{1}{2} \sqrt{\frac{\pi}{a}} \left(\frac{1}{2a} + v_0^2 \right) \left[\operatorname{erf} a^{\frac{1}{2}} (v_A - v_0) + \operatorname{erf} a^{\frac{1}{2}} v_0 \right] +$$

$$+ \frac{v_0}{2a} \exp(-av_0^2) - \frac{v_A + v_0}{2a} \exp\left\{ -a(v_A - v_0)^2 \right\}$$

and

$$\int_0^{v_A} v^2 \left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a(v + v_0)^2\right\}\right] dv = \frac{1}{2}\sqrt{\frac{\pi}{a}}\left(\frac{1}{2a} + v_0^2\right)$$

$$\left[\operatorname{erf} a^{\frac{1}{2}}(v_A - v_0) - \operatorname{erf} a^{\frac{1}{2}}(v_A + v_0) + 2 \operatorname{erf} a^{\frac{1}{2}} v_0\right] + \frac{v_0}{a} \exp(-av_0^2) -$$

$$- \frac{v_A + v_0}{2a} \exp\left\{-a(v_A - v_0)^2\right\} + \frac{v_A - v_0}{2a} \exp\left\{-a(v_A + v_0)^2\right\} .$$

For the contribution of the first part of the integral to the current, this gives

$$I_1 = \pi r^2 Ne \sqrt{\frac{a}{\pi}} \frac{1}{v_0}\left[\frac{1}{2}\sqrt{\frac{\pi}{a}}\left(\frac{1}{2a} + v_0^2\right)\left\{\operatorname{erf} a^{\frac{1}{2}}(v_A - v_0) - \operatorname{erf} a^{\frac{1}{2}}(v_A + v_0) +\right.\right.$$

$$\left. + 2 \operatorname{erf} a^{\frac{1}{2}} v_0\right\} + \frac{v_0}{a} \exp(-av_0^2) - \frac{v_A + v_0}{2a} \exp\left\{-a(v_A - v_0)^2\right\} +$$

$$\left. + \frac{v_A + v_0}{2a} \exp\left\{-a(v_A + v_0)^2\right\}\right] .$$

The ϵ integral for the second part is

$$\int_0^{\epsilon_A} \cos\epsilon \sin\epsilon \, d\epsilon = \frac{1}{2}\sin^2\epsilon_A = \frac{1}{2}\frac{\alpha}{\alpha + 1}\left(1 + \frac{v_A^2}{\alpha v^2}\right) ,$$

giving for I_2

$$I_2 = \pi r^2 Ne \sqrt{\frac{a}{\pi}} \frac{1}{v_0} \frac{\alpha}{\alpha + 1} \int_{v_A}^{\infty}\left(v^2 + \frac{v_A^2}{\alpha}\right)\left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a(v + v_0)^2\right\}\right] dv.$$

This is identical to the expression involved in the retarding form of the expression, except for the values of the constants. Hence, we have immediately,

$$I_2 = \pi r^2 \, Ne \sqrt{\frac{a}{\pi}} \frac{1}{v_0} \quad \frac{\alpha}{\alpha + 1} \left[\frac{1}{2} \sqrt{\frac{\pi}{a}} \left(\frac{1}{2a} + v_0^2 + \frac{v_A^2}{\alpha} \right) \left\{ \text{erf}\, a^{\frac{1}{2}} (v_A + v_0) - \right.\right.$$

$$\left. - \text{erf}\, a^{\frac{1}{2}} (v_A - v_0) \right\} + \frac{v_A + v_0}{2a} \exp\left\{ -a(v_A - v_0)^2 \right. -$$

$$\left.\left. - \frac{v_A - v_0}{2a} \exp\left\{ -a(v_A + v_0)^2 \right\} \right] .$$

For the total current $I_1 + I_2$, we now have

$$I = \pi r^2 \, Ne \sqrt{\frac{a}{\pi}} \frac{1}{v_0} \left[\sqrt{\frac{\pi}{a}} \left(\frac{1}{2a} + v_0^2 \right) \text{erf}\, a^{\frac{1}{2}} v_0 + \frac{v_0}{a} \exp(-av_0^2) - \right.$$

$$- \frac{1}{\alpha + 1} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{a}} \left(\frac{1}{2a} + v_0^2 - v_A^2 \right) \left[\text{erf}\, a^{\frac{1}{2}} (v_A + v_0) - \text{erf}\, a^{\frac{1}{2}} (v_A - v_0) \right] + \right.$$

$$\left.\left. + \frac{v_A + v_0}{2a} \exp\left[-a(v_A - v_0)^2 \right] - \frac{v_A - v_0}{2a} \exp\left[-a(v_A + v_0)^2 \right] \right\} \right] .$$

Defining parameters as for the retarding case,

$$\gamma^2 = av_0^2 = \frac{m}{2kT} v_0^2 , \qquad x^2 = av_A^2 = \alpha \frac{|eV|}{kT} .$$

We have finally

$$I = \pi r^2 \, Ne \frac{1}{\sqrt{\pi a}} \frac{1}{\gamma} \left[\frac{\sqrt{\pi}}{2} (1 + 2\gamma^2) \text{ erf } \gamma + \gamma \exp(-\gamma^2) - \right.$$

$$- \frac{1}{\alpha + 1} \left\{ \frac{\sqrt{\pi}}{2} \left(\frac{1}{2} + \gamma^2 - x^2 \right) \left[\text{erf}(x + \gamma) - \text{erf}(x - \gamma) \right] + \right.$$

$$\left.\left. + \frac{1}{2} (x + \gamma) \exp\left[-(x - \gamma)^2 \right] - \frac{1}{2} (x - \gamma) \exp\left[-(x + \gamma)^2 \right] \right\} \right] .$$

Note that the expression within the braces multiplied by $1/(\gamma + 1)$ is identical to the retarding expression in form, but that x has a slightly different definition.

It is also worth noting that the coefficient outside the whole expression contains the radius of the collector R in the retarding case, and the radius of the outer spherical surface r in the accelerating case.

6. LIMITING FORM OF THE EXPRESSIONS FOR NO-DRIFT

The expression for no-drift can be obtained from the equations derived here by taking the limit $v_0 \to 0$, that is, $\gamma \to 0$. This forms an important check on the expressions, as the equations for no-drift are relatively easily derived from first principles.

Considering first the retarding expression, we require

$$\lim_{\gamma \to 0} \frac{1}{\gamma}\left[\frac{\sqrt{\pi}}{2}\left(\frac{1}{2} + \gamma^2 - x^2\right)\left\{\operatorname{erf}(x+\gamma) - \operatorname{erf}(x-\gamma)\right\} + \frac{1}{2}(x+\gamma)\exp\left\{-(x-\gamma)^2\right\} - \right.$$

$$\left. - \frac{1}{2}(x-\gamma)\exp\left\{-(x+\gamma)^2\right\}\right] = \lim_{\gamma \to 0}\left[\frac{\sqrt{\pi}}{2}\left(\frac{1}{2} + \gamma^2 - x^2\right)\frac{\operatorname{erf}(x+\gamma) - \operatorname{erf}(x-\gamma)}{\gamma} + \right.$$

$$\left. + \frac{1}{2}x\,\frac{\exp\{-(x-\gamma)^2\} - \exp\{-(x+\gamma)^2\}}{\gamma} + \frac{1}{2}\exp\left\{-(x-\gamma)^2\right\} + \frac{1}{2}\exp\left\{-(x+\gamma)^2\right\}\right] =$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{1}{2} - x^2\right)\frac{4}{\sqrt{\pi}}\exp(-x^2) + \frac{1}{2}x\,4x\exp(-x^2) = 2\exp(-x^2) \quad .$$

Hence on substituting, we obtain

$$I = 2\pi R^2\, Ne\frac{1}{\sqrt{\pi a}}\exp(-x^2) \quad .$$

The accelerating expression can be dealt with in a similar manner and involves the same limiting expressions

$$\lim_{\gamma \to 0}\frac{1}{\gamma}\left[\frac{\sqrt{\pi}}{2}(1 + 2\gamma^2)\operatorname{erf}\gamma + \gamma\exp(-\gamma^2) - \frac{1}{\alpha + 1}\left\{\frac{\sqrt{\pi}}{2}\left(\frac{1}{2} + \gamma^2 - x^2\right)\right.\right.$$

$$\left[\operatorname{erf}(x+\gamma) - \operatorname{erf}(x-\gamma)\right] + \frac{1}{2}(x+\gamma)\exp\left[-(x-\gamma)^2\right] -$$

$$\left.\left. - \frac{1}{2}(x-\gamma)\exp\left[-(x+\gamma)^2\right]\right\}\right] = \lim_{\gamma \to 0}\left[\frac{\sqrt{\pi}}{2}\,\frac{\operatorname{erf}\gamma}{\gamma} + \exp(-\gamma^2) - \right.$$

$$- \frac{1}{\alpha + 1} \left\{ \frac{\sqrt{\pi}}{2} \left(\frac{1}{2} - x^2 \right) \frac{\operatorname{erf}(x + \gamma) - \operatorname{erf}(x - \gamma)}{\gamma} + \frac{1}{2} x \, \frac{\exp[-(x - \gamma)^2] - \exp[-(x - \gamma)^2]}{\gamma} + \right.$$

$$\left. + \frac{1}{2} \exp \left[-(x - \gamma)^2 \right] + \frac{1}{2} \exp \left[-(x + \gamma)^2 \right] \right\} \Bigg] = 2 - \frac{1}{\alpha + 1} \left\{ \frac{\sqrt{\pi}}{2} \left(\frac{1}{2} - x^2 \right) \frac{4}{\sqrt{\pi}} \exp(-x^2) + \right.$$

$$\left. + 2x^2 \exp(-x^2) + \exp(-x^2) \right\} = 2 \left[1 - \frac{1}{\alpha + 1} \exp(-x^2) \right] \quad .$$

Hence, for the current we have

$$I = 2\pi r^2 \, Ne \frac{1}{\sqrt{\pi a}} \left[1 - \frac{1}{\alpha + 1} \exp(-x^2) \right] \quad .$$

Note that for $x = 0$ (that is, $V = 0$) both expressions reduce to the same equation, namely,

$$I = 2\pi R^2 \, Ne \frac{1}{\sqrt{\pi a}} \quad .$$

7. THE SHEATH

Referring back to the calculations we have made, we see that in setting up our original integral we considered particles crossing a spherical surface of radius r. In defining the limits of integration we determined which of these particles is capable of reaching a concentric spherical surface of radius R where $R < r$ when there is a potential difference V between the two surfaces. We also considered only inward-travelling particles in setting up the limits of integration. This means that we preclude the possibility of the potential on the inner sphere influencing an outward-travelling particle at radius r in such a way as to reverse its outward motion so that it can eventually reach the sphere radius R. This is equivalent to supposing that the electric field due to the potential on the inner sphere does not extend beyond a distance r. This, then, is very close to Langmuir's concept of the sheath radius.

In deriving the above expressions, we have assumed that we can equate the potential to zero over the sphere radius r, that is, that no particle outside this spherical surface can experience any force from the influence of the field. We have assumed that the velocity distribution of the inward-travelling particles at radius r follows Maxwell's law. (The velocity distribution of the outward-travelling particles is immaterial, as they do not enter into the calculation.) In deriving the expressions for the limits, we used the energy and momentum of the particle at

the two surfaces only and applied the conservation laws. We, therefore, implicitly assumed that there are no particle collisions between the two surfaces. Also in applying these results, it is assumed that the potential distribution between the two surfaces is such that the radial velocity of the particle nowhere becomes zero, that is, that there is no potential barrier which the particle cannot surmount. This is worth considering in a little more detail as it has important applications in considering a multi-grid sensor.

The argument follows closely that used to obtain the limits of the integral. Suppose we have a particle at the sheath edge at radius r_i travelling with velocity v_i at an angle ϵ_i to the radius vector, and suppose that at some arbitrary value of r, $R < r < r_i$, the velocity and angle are v and ϵ. For the particle to reach the collector, not only must $\sin\epsilon_c$ (the value at the collector) be less than unity, but $\sin\epsilon$ must be less than unity for all values of r within the sheath. This means, from the arguments given earlier, that

$$\sin^2\epsilon_i < \frac{r^2}{r_i^2}\left(1 - \frac{2eV}{mv_i^2}\right) .$$

This condition will apply in both retarding and accelerating cases if e and V are considered to contain the sign as well as the magnitude of charge and voltage. But the relation between the limiting values of ϵ_i and v_i, if the particle is to be collected at all, satisfy the equation

$$\sin^2\epsilon_i = \frac{R^2}{r_i^2}\left(1 - \frac{2eV_c}{mv_i^2}\right) ,$$

where V_c is the potential of the collector. Hence, eliminating v_i

$$\sin^2\epsilon_i < \frac{r^2}{r_i^2}\left[1 + \frac{V}{V_c}\left(\frac{r_i^2 \sin^2\epsilon_i}{R^2} - 1\right)\right]$$

which on rearranging gives

$$\sin^2\epsilon_i < \left(1 - \frac{V}{V_c}\right)\Bigg/\left(\frac{r_i^2}{r^2} - \frac{V}{V_c}\frac{r_i^2}{R^2}\right) .$$

The most stringent case to be satisfied occurs for the accelerating condition when ϵ_i can assume the value $\pi/2$. Hence, the condition is satisfied for all cases if

$$\left(1 - \frac{V}{V_c}\right) \bigg/ \left(\frac{r_i^2}{r^2} - \frac{V}{V_c}\frac{r_i^2}{R^2}\right) > 1$$

which on rearranging gives

$$\frac{V}{V_c} > \frac{r_i^2 - r^2}{r_i^2 - R^2}\frac{R^2}{r^2} \quad .$$

Since V_c and V may be either positive or negative, this result expressed as a condition for V becomes

$$|V| > \frac{r_i^2 - r^2}{r_i^2 - R^2}\frac{R^2}{r^2}|V_c| \quad .$$

Note that, since we defined the potential in the plasma outside the sheath to be zero, and also $R < r < r_i$, the limiting condition implies that $0 < |V| < |V_c|$.

8. POWER SERIES EXPANSION OF THE ACCELERATING EQUATION

A useful expression can be obtained by expanding the expression for the accelerating case as a Taylor series in powers of V, valid near the origin V = 0. The first three terms in the series are

$$I = I_0 + \left(\frac{dI}{dV}\right)_0 V + \frac{1}{2}\left(\frac{d^2I}{dV^2}\right)_0 V^2 + \ldots \quad .$$

where the subscript 0 means the value of the quantity concerned at V = 0. The expression derived earlier gives I as a function of x rather than of V. A complication arises now, since we defined x using the magnitude of eV —

$$x^2 = \alpha\frac{|eV|}{kT} \quad .$$

It is convenient to calculate the derivative (dI/dV) as $(dI/dx)(dx/dV)$, but in differentiating the equation which relates x to V, the sign of dx/dV is ambiguous. This is readily resolved by noting that the sign of the derivative is the same as the sign of e, that is, it is positive for positively charged particles, and negative for negatively charged particles. The following expressions are now quite straightforward to obtain

$$I_0 = \pi R^2 \, Ne \frac{1}{\sqrt{\pi a}} \, \frac{1}{\gamma} \left[\frac{\pi}{2} \, (1 + 2\gamma^2) \, \text{erf} \, \gamma + \gamma \, \exp \, (-\gamma^2) \right] \quad .$$

$$\left(\frac{dI}{dV}\right)_0 = \mp \pi R^2 \, Ne \frac{1}{\sqrt{\pi a}} \, \frac{1}{\gamma} \frac{\sqrt{\pi}}{2} \frac{e}{kT} \, 2 \, \text{erf} \, \gamma \quad . \qquad \begin{matrix} \text{Positive particles} \\ \text{Negative particles} \end{matrix}$$

$$\left(\frac{d^2I}{dV^2}\right)_0 = \pi R^2 \, Ne \frac{1}{\sqrt{\pi a}} \frac{1}{\gamma} \left(\frac{e}{kT}\right)^2 \alpha \, (-4\gamma) \, \exp \, (-\gamma^2) \quad .$$

Hence, the expansion for I becomes

$$I = \pi R^2 \, Ne \frac{1}{\sqrt{\pi a}} \, \frac{1}{\gamma} \left[\frac{\sqrt{\pi}}{2} \, (1 + 2\gamma^2) \right] \text{erf} \, \gamma - \gamma \, \exp \, (-\gamma^2) \mp$$

$$\mp \frac{\sqrt{\pi}}{2} \, 2 \left\{ \text{erf} \, \gamma \right\} \left(\frac{eV}{kT}\right) - \frac{1}{2} \alpha \;\; \gamma \left\{ \exp \, (-\gamma^2) \right\} \left(\frac{eV}{kT}\right)^2 \ldots . \Big] \quad ,$$

or with some rearrangement of the constants,

$$I = \pi R^2 \, Ne \, v_0 \left[\left(1 + \frac{1}{2\gamma^2}\right) \, \text{erf} \, \gamma + \frac{2}{\sqrt{\pi}} \, \frac{1}{2\gamma} \, \exp \, (-\gamma^2) \mp \frac{1}{\gamma^2} \left\{ \text{erf} \;\; \gamma \right\} \left(\frac{eV}{kT} \right) - \right.$$

$$\left. - \frac{1}{2} \, \alpha \frac{2}{\sqrt{\pi}} \, \frac{1}{2\gamma} \left\{ \exp \, (-\gamma^2) \right\} \left(\frac{eV}{kT}\right)^2 \ldots \right] \quad .$$

On referring back to the complete expression for the accelerating case we note that it contains the "sheath radius" r rather than the collector radius R. We also recall that r is the radius of the spherical surface on which we can assume $V = 0$ and this will presumably be a function of V, the voltage on the collector. Hence, we do not have I explicitly expressed as a function of V for the accelerating case

until we have r as a function of V. However, in the series expansion, the collector radius R appears outside the expression and r, which occurs in α, does not appear until the term in V^2. Therefore, the linear approximation (the first two terms) is independent of the behavior of the sheath. Unfortunately, we cannot estimate how large the voltage must be before the linear approximation breaks down, since the next higher order term does depend upon the behavior of the sheath.

It is readily verified from the retarding expression that the values of I_0 and $(dI/dV)_0$ are identical to those obtained from the accelerating expression. Thus, the curve of I versus V is continuous and so is its derivative at V = 0, that is, as we go from retarding to accelerating. There is, however, a discontinuity in the second derivative $(d^2I/dV^2)_0$.

9. LIMITING FORM OF THE EXPRESSIONS FOR LARGE DRIFT VELOCITIES

The behavior of the expressions for large drift velocities can be obtained from the power series expansion. This expansion was derived above for the accelerating case, and since $v_0 = \gamma/\sqrt{a}$, this may be written

$$I = \pi R^2 Ne\frac{1}{\sqrt{a}}\left[\left(\gamma + \frac{1}{2\gamma}\right) \text{erf}\, \gamma + \frac{2}{\sqrt{\pi}} \frac{1}{2} \exp(-\gamma^2) \mp \frac{1}{\gamma} \left\{\text{erf}\, \gamma\right\}\left(\frac{eV}{kT}\right) - \right.$$

$$\left. -\frac{1}{2}\alpha \frac{2}{\sqrt{\pi}} \frac{1}{2} \left\{\exp(-\gamma^2)\right\}\left(\frac{eV}{kT}\right)^2 \ldots\right] .$$

If we now consider large drift velocities then we need the case for large γ. The second term in the expression rapidly goes to zero since it varies as $\exp(-\gamma^2)$. The error function becomes asymptotically unity for large values of the argument, thus the constant term in the expression becomes equal to γ. The term containing $(eV/kT)^2$ is difficult to deal with since it contains α. However, we note that it also contains $\exp(-\gamma^2)$ which tends to zero very rapidly for large γ. For smaller values of (eV/kT), where the second and higher order terms can certainly be neglected, we have

$$I = \pi R^2 Ne\, v_0 \left[1 \mp \frac{1}{\gamma^2} \frac{eV}{kT}\right] .$$

Thus, the linearity of the accelerating characteristic is not destroyed by large rocket velocities. In fact, the linear approximation may be better at larger rocket velocities, since the linear term decreases as $(\text{erf}\, \gamma)/\gamma$, whereas, the square term

decreases as $\exp(-\gamma^2)$. The term containing V^2 also has α in its coefficient, and the variation of α with γ has not been calculated. However, it seems unlikely that α will increase with sufficient speed with γ to offset the extremely rapid decrease due to the exponential factor. Consequently the square term becomes relatively less important in comparison with the linear term.

The slope (dI/dV) decreases as $(1/v_0^2)$ and thus, in the limit, becomes zero. The current is now independent of the voltage and, as we would expect from elementary considerations, has the value

$$I_0 = \pi R^2 \, Ne \, v_0 \quad .$$

Carrying through a similar expansion for the retarding case gives

$$I = \pi R^2 \, Ne \frac{1}{\sqrt{a}} \left[\left(\gamma + \frac{1}{2\gamma}\right) \operatorname{erf} \gamma + \frac{2}{\sqrt{\pi}} \frac{1}{2} \exp(-\gamma^2) \mp \frac{1}{\gamma} \left\{ \operatorname{erf} \gamma \right\} \left(\frac{eV}{kT}\right) - \right.$$

$$\left. - \frac{1}{2} \frac{2}{\sqrt{\pi}} \frac{1}{2} \left\{ \exp(-\gamma^2) \right\} \left(\frac{eV}{kT}\right)^2 + \ldots \right] \quad .$$

Note that this expression (valid for all γ) tends to the series expansion for $\exp(eV/kT)$ for $\gamma \to 0$ as it should. For large γ the constant term and the term in (eV/kT) are identical to those for the accelerating case and, hence, the continuity of I and (dI/dV) at $V = 0$ (the changeover from retarding to accelerating) is preserved at high drift velocities. The exponential form, however, is destroyed. The ratio of the coefficients of V^2 and V in the expansion is

$$\frac{1}{2\sqrt{\pi}} \frac{e}{kT} \frac{\gamma \exp(-\gamma^2)}{\operatorname{erf} \gamma} \quad .$$

As v_0 becomes large this tends to zero as $\gamma \exp(-\gamma^2)$ and the second order term diminishes very rapidly with increasing drift. The retarding characteristic also becomes linear, and from the continuity noted above the straight line of the accelerating characteristic is extended into the retarding region. For a fixed value of v_0, if the voltage is increased sufficiently, x will eventually dominate over γ again in the expression and the curve departs from linearity, and the exponential form is regained.

10. THE MULTI-GRID SENSOR

To extend the analysis for a case where the incoming particles flow from the ambient plasma through a set of concentric spherical grids to a central collector, it is necessary to take the equation for I (p. 11)

$$I = 2\pi r^2 \, N e \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \frac{1}{v_0} \int_v \int_\epsilon v^2 \left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a\,(v + v_0)^2\right\}\right] \cos\epsilon \sin\epsilon \, d\epsilon \, dv.$$

and perform the final integrations according to the limits on v and ϵ, given by energy and momentum equations in transition from grid to grid.

Consider a particle at grid i of radius r_i with velocity v_i at an angle ϵ_i to the inward radius vector arriving at grid i + 1 radius r_{i+1} ($r_{i+1} < r_i$) with velocity v_{i+1} at an angle ϵ_{i+1}. If potentials on the two grids are V_i and V_{i+1}, then from energy considerations

$$\frac{1}{2} m v_i^2 = \frac{1}{2} m v_{i+1}^2 + e(V_{i+1} - V_i) \quad .$$

Defining a velocity u_i by the relation $\frac{1}{2} m u_i^2 = eV_i$, this equation becomes

$$v_i^2 = v_{i+1}^2 + u_{i+1}^2 - u_i^2 \quad .$$

From angular momentum considerations,

$$r_i v_i \sin\epsilon_i = r_{i+1} v_{i+1} \sin\epsilon_{i+1} \quad .$$

The particle will just enter the grid i + 1 if it arrives at the grid travelling tangentially, that is, if $\sin\epsilon_{i+1} \leq 1$, that is, if

$$v_{i+1} \geq v_i \frac{v_i r_i}{r_{i+1}} \sin\epsilon_i \quad ,$$

which, on substituting in the energy relationship, gives

$$v_i^2 + u_i^2 - u_{i+1}^2 \geq v_i^2 \frac{r_i^2}{r_{i+1}^2} \sin^2\epsilon_i \quad .$$

Hence, for a given value of v_i, the particle passes through grid i + 1 if

$$\sin^2 \epsilon_i \le \frac{r_{i+1}^2}{r_i^2}\left[1 + \frac{u_i^2 - u_{i+1}^2}{v_i^2}\right] \quad .$$

We note that the above relation is satisfied for all ϵ_i $(0 \le \epsilon_i \le \frac{\pi}{2})$ if

$$\frac{r_{i+1}^2}{r_i^2}\left[1 + \frac{u_i^2 - u_{i+1}^2}{v_i^2}\right] \ge 1$$

or

$$v_i^2 \le (u_i^2 - u_{i+1}^2)\,\frac{r_{i+1}^2}{r_i^2 - r_{i+1}^2} \quad ,$$

which can only apply for $u_i^2 > u_{i+1}^2$, that is, accelerating potentials from grid i to grid i + 1 since $v_i^2 \ge 0$. For $v_i^2 > (u_i^2 - u_{i+1}^2)\alpha_{i+1}$

$$\left(\alpha_{i+1} = \frac{r_{i+1}^2}{r_i^2 - r_{i+1}^2}\right) \quad .$$

Limits on ϵ_i are then given by the limits on v_i of $v_{i(min)}$ and $v_{i(max)}$, thus

$$\frac{r_{i+1}^2}{r_i^2}\left[1 + \frac{u_i^2 - u_{i+1}^2}{v_{i(min)}^2}\right] \ge \sin^2 \epsilon_i \ge \frac{r_{i+1}^2}{r_i^2}\left[1 + \frac{u_i^2 - u_{i+1}^2}{v_{i(max)}^2}\right] \quad .$$

For retarding potentials $u_i^2 < u_{i+1}^2$, a lower limit is imposed on v_i by the condition that $\sin^2 \epsilon_i \ge 0$, hence

$$\frac{r_{i+1}^2}{r_i^2}\left[1+\frac{u_i^2-u_{i+1}^2}{v_i^2}\right] \geq 0$$

or

$$v_i^2 \geq u_{i+1}^2 - u_i^2 \quad .$$

The set of limits,

$$u_i^2 - u_{i+1}^2 \geq 0 \qquad \text{(retarding)}$$

$$\left.\begin{aligned} &\left[u_{i+1}^2 - u_i^2\right] \leq v_i^2 \leq \infty \\ &0 \leq \sin^2\epsilon_i \leq \frac{r_{i+1}^2}{r_i^2}\left[1+\frac{u_i^2-u_{i+1}^2}{v_i^2}\right] \end{aligned}\right\} ,$$

$$u_i^2 - u_{i+1} < 0 \qquad \text{(accelerating)}$$

$$\left.\begin{aligned} &0 < v_i^2 \leq (u_i^2 - u_{i+1}^2)\,\alpha_{i+1} \\ &0 \leq \sin^2\epsilon_i \leq 1 \end{aligned}\right\} \qquad \left.\begin{aligned} &(u_i^2 - u_{i+1}^2)\,\alpha_{i+1} < v_i^2 \leq \infty \\ &0 \leq \sin^2\epsilon_i \leq \frac{r_{i+1}^2}{r_i^2}\left[1+\frac{u_i^2-u_{i+1}^2}{v_i^2}\right] \end{aligned}\right\} ,$$

on v_i and ϵ_i, imposed by the condition that the particle reach the $i + 1^{th}$ grid, can be extended to the $i + k^{th}$ grid, since the velocity with which the particle emerges at this grid is a function only of v_i, ϵ_i, u_i, r_i, u_{i+k}, and r_{i+k}.

Hence for the k^{th} grid

$$\sin^2\epsilon_i \leq \frac{r_{i+k}}{r_i^2}\left[1+\frac{u_i^2-u_{i+k}^2}{v_i^2}\right] \quad .$$

For $i = 0$, let $r_0 = r$, and $u_0 = 0$. Hence, for the i^{th} grid, $\sin^2\epsilon \leq \frac{r_1^2}{r^2}\left[1-\frac{u_i^2}{v^2}\right]$ dropping the 0^{th} subscript.

The effects of the various grids in limiting v and ϵ (the initial particle parameters at grid 0) can now be compared and the integral I performed according to the way each grid (i = 1, 2, 3, etc.) limits the values of v and ϵ. Since the lower limit on ϵ_i is always zero, the ϵ integration may be performed prior to discussing the specific applicable limits on v, writing $r^2/r_i^2 = \beta_i^2$, we have

$$\int_0^{\epsilon_i} \cos\epsilon \sin\epsilon \, d\epsilon = \frac{1}{2}\sin^2\epsilon_i = \frac{1}{2\beta_i^2}\left[1 - \frac{u_i^2}{v^2}\right] .$$

Hence, we may write for I,

$$I = 2\pi r^2 \, Ne \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \frac{1}{v_0} \int_{v_o} \frac{v^2}{2\beta_i^2}\left[1 - \frac{u_i^2}{v^2}\right]\left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a(v + v_0)^2\right\}\right] dv.$$

The limits imposed by v^2 on $\sin^2\epsilon$ can be illustrated by plotting the limiting curve

$$\sin^2\epsilon = \frac{r_i^2}{r^2}\left[1 - \frac{u_i^2}{v^2}\right] = \frac{1}{\beta_i^2}\left[1 - \frac{u_i^2}{v^2}\right]$$

for each grid of radius r_i with potential V_i ($u_i^2 = 2eV_i/m$).

Various potentials will be considered for a four-grid case, to illustrate the progression of limits on the I integral. These are so chosen to include all possible I integrals but do not cover all possible potential differences between the various grids. The resulting equations will then be extended to cover the general case for N grids.

Consider the i^{th} and the k^{th} grid, where $k > i$.

$$\mathrm{Sin}^2\epsilon = \frac{1}{\beta_i^2}\left[1 - \frac{u_i^2}{v_i^2}\right]$$

and

$$\sin^2\epsilon = \frac{1}{\beta_k^2}\left[1 - \frac{u_k^2}{v^2}\right] .$$

The intersection point of these two curves is given by

$$\left.\begin{aligned} v^2 = v_{ik}^2 &= \frac{\beta_k^2 u_i^2 - \beta_i^2 u_k^2}{\beta_k^2 - \beta_i^2} \\ \sin^2\epsilon = \sin^2\epsilon_{ik} &= \frac{u_i^2 - u_k^2}{\beta_k^2 u_i^2 - \beta_i^2 u_k^2} \end{aligned}\right\} \qquad \begin{aligned} \beta_i^2 &= \frac{r^2}{r_i^2} \\ u_i^2 &= \frac{2eV_i}{m} \end{aligned}$$

and for the intersection point to lie within the domain of the integral (ϵ, v) then

$$v_{ik}^2 \geq 0$$

and

$$0 \leq \sin^2\epsilon_{ik} \leq 1 .$$

Since $\beta_k^2 > \beta_i^2 > 1$ then

$$u_k^2 > \frac{u_k^2 \beta_i^2}{\beta_k^2} > \frac{u_k^2 (\beta_i^2 - 1)}{\beta_k^2 - 1} .$$

Giving the restrictions on u_i^2 of

$$\left.\begin{aligned} & u_i^2 \geq u_k^2 && \text{for } u_k^2 \geq 0 \\ & \text{and} \\ & u_i^2 \geq u_k^2 \frac{\beta_i^2 - 1}{\beta_k^2 - 1} && \text{for } u_k^2 < 0 \end{aligned}\right\} \text{ for real } \epsilon_{ik}, v_{ik} \text{ and where } i < k.$$

Before turning to specific cases it is of interest to note that for the outer grid radius r, defined to be at zero potential,

$$\beta_0^2 = \frac{r^2}{r^2} = 1 \qquad \text{and } u_0^2 = 0.$$

Hence,

$$\left.\begin{aligned} v_{0k}^{\,2} &= \frac{-u_k^2}{\beta_k^2 - 1} \\ \sin^2 \epsilon_{0k} &= 1 \end{aligned}\right\}$$

From which it can be seen that, for the outer grid to contribute to the I integral, the k^{th} grid potential must be such at $u_k^2 < 0$. Hereafter, the grid k having $u_k^2 < 0$, that is, $eV_k < 0$ or the sign of the grid potential being opposite to that of the particle charge, will be termed an accelerating grid. Conversely, one having $u_k^2 \geq 0$, will be termed a retarding grid.

Since the above equations define an upper limit to $\sin^2\epsilon$, the limits of integration lie below the limiting curves.

Taking a configuration of four grids with the radii having the ratios

$$r^2 : r_1^2 : r_2^2 : r_3^3 = 1 : 0.8 : 0.4 : 0.2,$$

hence,

$$\beta_1^2 = \frac{r^2}{r_1^2} = 1.25, \quad \beta_2^2 = 2.5, \quad \text{and} \quad \beta_3^2 = 5.0.$$

We can now investigate the successive limits on the I integral by applying various combinations of voltages to grids 1, 2, and 3.

(a) $u_3^2 > u_2^2 > u_1^2 > 0$

The limiting curves for $u_3^2 = 6$, $u_2^2 = 4$, and $u_1^2 = 3$ are shown in Figure 5. Since $u_i^2 \not> u_k^2$, no curves intersect and the limits on v go from point A to infinity

along the curve defined by $u_3^2 = 6$, that is, the potential on grid 3 defines the limit on I and becomes the only active grid (that is, the only grid limiting the current flow). The intersection of $u_3^2 = 6$ with $\sin^2 \epsilon_0 = 0$ is given by $v^2 = u_3^2 = 6$, and I has the value

$$I = 2\pi r^2 \, Ne\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \frac{1}{v_0} \int_{u_3}^{\infty} \frac{v^2}{2\beta_3^2} \left[1 - \frac{u_3^2}{v^2}\right] \left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a(v + v_0)^2\right\}\right] dv.$$

Or, writing

$$2\pi Ne\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \frac{1}{v_0} = P \ , \qquad \left[\exp\left\{-a(v - v_0)^2\right\} - \exp\left\{-a(v + v_0)^2\right\}\right] = E(v) \ ,$$

and substituting

$$\beta_3^2 = \frac{r^2}{r_3^2}$$

gives

$$I = P \int_{u_3}^{\infty} r_3^2 \frac{v^2}{2} \left[1 - \frac{u_3^2}{v^2}\right] E(v) \, dv \ . \tag{1}$$

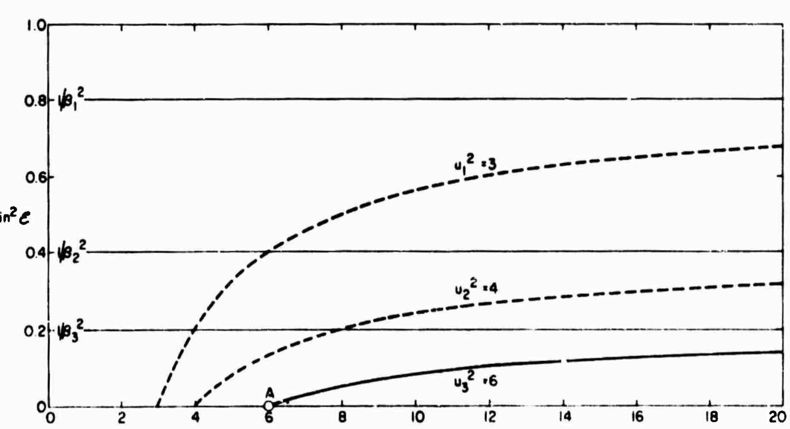

Figure 5. Limits of Integration for $u_3^2 > u_2^2 > u_1^2 > 0$

(b) $u_1^2 > u_2^2 > u_3^2 > 0$

Figure 6 shows curves for $u_3^2 = 1$, $u_2^2 = 2$, and $u_1^2 = 3$. The condition for real intersection, $u_i^2 \geq u_k^2$, is now fulfilled by grids 1, 2, and 3, but not by grid 0. The three intersection points are given by $v_{12}^2 = 4$, $v_{13}^2 = 11/3$, and $v_{23}^2 = 3$ at B_1, B, and B_2 respectively.

The limits on v_2 now commence at A with the intersection of $\sin^2 \epsilon = 0$ and the most retarding grid (in this case grid 1) intersecting at $v^2 = 3$. The limiting curve is then given by $u_1^2 = 3$ up to the smaller of the two values v_{12}^2 and v_{13}^2. Here, $v_{13}^2 = B = 11/3$ has the smallest v^2, and for $v^2 > 11/3$, the limiting curve becomes $u_3^2 = 1$. The integral I is now the sum of two parts—from A to B along $u_1^2 = 3$ and from B to ∞ along $u_3^2 = 1$, and grids 1 and 3 are active grids.

$$I = P\left\{\int_{u_1}^{v_{13}} r_1^2 \frac{v^2}{2} \left[1 - \frac{u_1^2}{v^2}\right] E(v)\, dv + \int_{v_{13}}^{\infty} r_3^2 \frac{v^2}{2} \left[1 - \frac{u_3^2}{v^2}\right] E(v)\, dv\right\}, \quad (2)$$

where,

$$v_{13}^2 = \frac{r_1^2 u_1^2 - r_3^2 u_3^2}{r_1^2 - r_3^2} \quad .$$

It is worth noting that in both Eq. 1 and Eq. 2, where the active grids are retarding, the expression for I is independent of r, the outer grid radius.

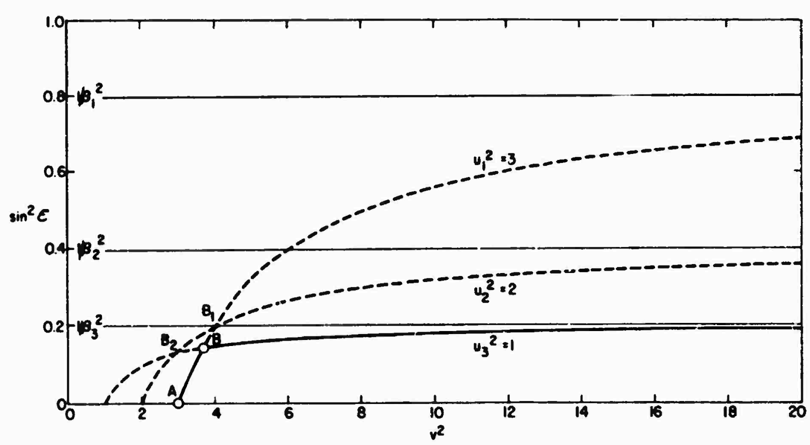

Figure 6. Limits of Integration for $u_3^2 > u_2^2 > u_1^2 > 0$

(c) $0 > u_1^2 > u_2^2 > u_3^2$

Figure 7 has $u_3^2 = -8$, $u_2^2 = -6$, and $u_1^2 = -1.5$. The condition now becomes $u_i^2 > u_k^2 \beta_i^2 / \beta_k^2$ for real intersections. This condition is only met by $i = 0$, $k = 1$, 2, and 3, where $v_{01}^2 = 6$, $v_{02}^2 = 4$, and $v_{03}^4 = 2$, as shown in the figure at B_2, B_1, and B respectively.

The I integral is now defined by $\sin^2 \epsilon = 1$ up to the smallest of v_{01}, v_{02}, and v_{03}, in this instance, v_{03}. For $v^2 > v_{03}^2$ is limited by the curve $u_3^2 = -8$, making grids 0 and 3 the only active grids. The case where the integral is limited by $\sin^2 \epsilon = 1$ is rather a special case, since the limiting case is given by $u_0^2 = 0$ and is independent of v. Returning to the ϵ integral, when $\epsilon_i = \pi/2$

$$\int_0^{\epsilon_i} \cos \epsilon \sin \epsilon \, d\epsilon = \frac{1}{2} \quad .$$

Hence, in general,

$$I_{\sin^2 \epsilon = 1} = P \int_0^{v_{0k}} r^2 \frac{v^2}{2} E(v) \, dv \quad ,$$

and, in this case,

$$I = P \left\{ \int_0^{v_{03}} r^2 \frac{v^2}{2} E(v) \, dv + \int_{v_{03}}^{\infty} r_3^2 \frac{v^2}{2} \left[1 - \frac{u_3^2}{v^2} \right] E(v) \, dv \right\} , \qquad (3)$$

where

$$v_{03}^2 = \frac{-r_3^2 u_3^2}{r^2 + r_3^2} \quad .$$

Since the 'most retarding grid' in this case is in fact an accelerating grid ('least accelerating') the current now becomes dependent on the outer grid radius r.

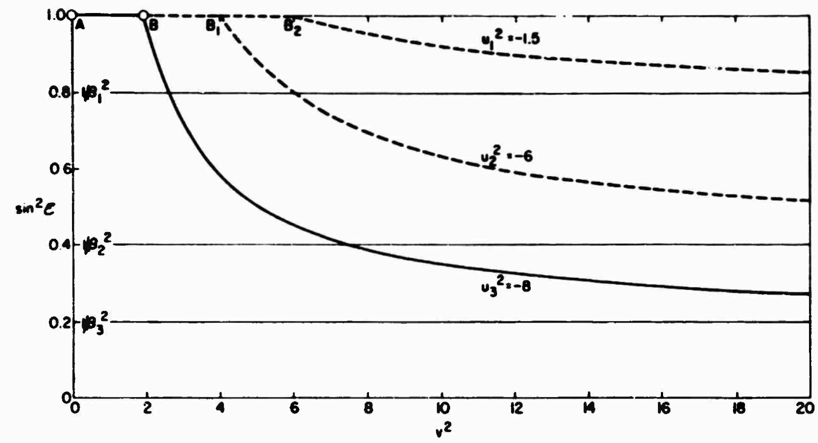

Figure 7. Limits of Integration for $0 > u_1^2 > u_2^2 > u_3^2$

(d) $0 > u_1^2 > u_2^2 > u_3^2$

Where Figure 8 now shows $u_3^2 = -8$, $u_2^2 = -6$, and $u_1^2 = -0.25$, $\sin^2\epsilon = 1$ ($u_0^2 = 0$) is the limiting curve up to B, the smallest intercept of $v_{03}^2 = 2(B_1)$, $v_{02}^2 = 4(B_2)$ and $V_{01}^2 = 1$ (B), making grid 0 an active grid. The limit is then given by $u_1^2 = -0.25$ up to C, the smallest intercept of $v_{12}^2 = 5.5$ (C_1) and $v_{13}^2 = 7/3$ (C). The final portion is from C to ∞ along $u_3^2 = -8$ making grids 1 and 3 the active grids, and

$$I = P\left\{\int_0^{v_{01}} r^2 \frac{v^2}{2} E(v)\,dv + \int_{v_{01}}^{v_{13}} r_1^2 \frac{v^2}{2}\left[1 - \frac{u_1^2}{v^2}\right] E(v)\,dv + \right. \tag{4}$$

$$\left. + \int_{v_{13}}^{\infty} r_3^2 \frac{v^2}{2}\left[1 - \frac{u_3^2}{v^2}\right] E(v)\,dv\right\} ,$$

where

$$v_{01}^2 = \frac{-r_1^2 u_1^2}{r^2 - r_1^2} \quad \text{and} \quad v_{13}^2 = \frac{r_1^2 u_1^2 - r_3^2 u_3^2}{r_1^2 - r_3^2} .$$

We will finally take two examples where all three grids become active and the outer grid is active or not according to whether all active grids are accelerating or not.

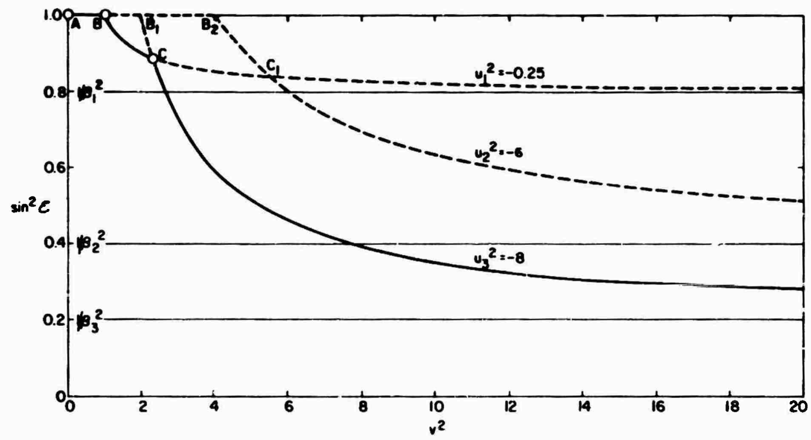

Figure 8. Limits of Integration for $0 > u_1^2 > u_2^2 > u_3^2$

(e) $u_1^2 > u_2^2 > 0 > u_3^2$

Illustrated in Figure 9 are $u_3^2 = -8$, $u_2^2 = 2$, and $u_1^2 = 3$. For the first lower limit on v, we examine retarding grids and determine the greatest u_i^2, that is, $u_1^2 = 3$, giving point A at $V^2 = u_1^2 = 3$. From A the limiting curve is $u_1^2 = 3$ up to the smallest of $v_{12}^2 = 4$(B) and $v_{13}^2 = 20/3$ (B_1). The integral from B is limited by $u_2^2 = 2$ up to the intercept $v_{23}^2 = 12$ (C), and from C to ∞ along $u_3^2 = -8$ giving

$$I = P\left\{\int_{u_1}^{v_{12}} r_1^2 \frac{v^2}{2}\left[1 - \frac{u_1^2}{v^2}\right]E(v)\,dv + \int_{v_{12}}^{v_{23}} r_2^2 \frac{v^2}{2}\left[1 - \frac{u_2^2}{v^2}\right]E(v)\,dv + \right. \tag{5}$$

$$\left. + \int_{v_{23}}^{\infty} r_3^2 \frac{v^2}{2}\left[1 - \frac{u_3^2}{v^2}\right]E(v)\,dv\right\} ,$$

where

$$v_{12}^2 = \frac{r_1^2 u_1^2 - r_2^2 u_2^2}{r_1^2 - r_2^2} \qquad \text{and} \qquad v_{23}^2 = \frac{r_2^2 u_2^2 - r_3^2 u_3^2}{r_2^2 - r_3^2} .$$

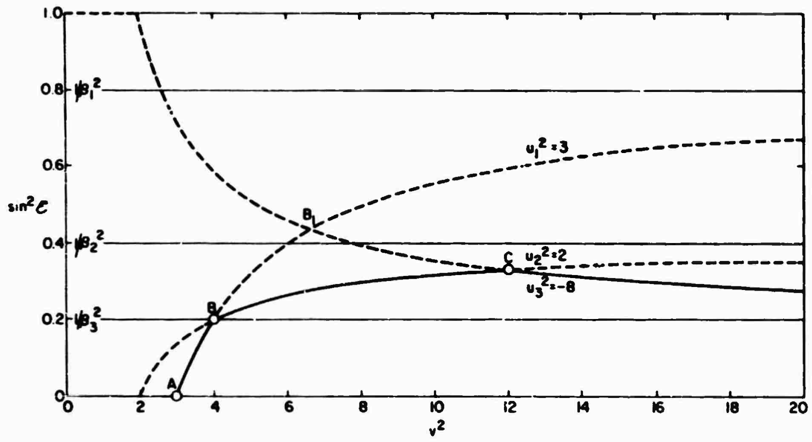

Figure 9. Limits of Integration for $u_1^2 > u_2^2 > 0 > u_3^2$

(f) $0 > u_1^2 > u_2^2 > u_3^2$

Finally, Figure 10 has $u_3^2 = -32$, $u_2^2 = -6$, and $u_1^2 = -0.25$. Because all grids are accelerating, we compare $v_{01}^2 = 1$ (B), $v_{02}^2 = 4$ (B_1), and $v_{03}^2 = 8$ (B_2), and find that grid 1 has the smallest intercept to give first integral term along $\sin^2 \epsilon = 1$ from A to B. The second term is then limited by $u_1^2 = -0.25$ from B to the smallest of $v_{12}^2 = 5.5$ (C) and $v_{13}^2 = 31/3$ (C_1), giving v_{12} as the upper limit of the second term and the lower limit of the third. The next term is then limited by $u_2^2 = -6$ up to $v_{23}^2 = 20$ (D) and the last term from D to ∞ by $u_3^2 = -32$. Whence,

$$I = P\left\{\int_0^{v_{01}} r^2 \frac{v^2}{2} E(v)\,dv + \int_{v_{01}}^{v_{12}} r_1^2 \frac{v^2}{2}\left[1 - \frac{u_1^2}{v^2}\right] E(v)\,dv + \int_{v_{12}}^{v_{23}} r_2^2 \frac{v^2}{2}\left[1 - \frac{u_2^2}{v^2}\right] E(v)\,dv + \int_{v_{23}}^{\infty} r_3^2 \frac{v^2}{2}\left[1 - \frac{u_3^2}{v^2}\right] E(v)\,dv\right\} \tag{6}$$

where

$$v_{ij}^2 = \frac{r_i^2 u_i^2 - r_j^2 u_j^2}{r_i^2 - r_j^2} \quad ; \qquad i = 0, 1, 2; \qquad j = i + 1 .$$

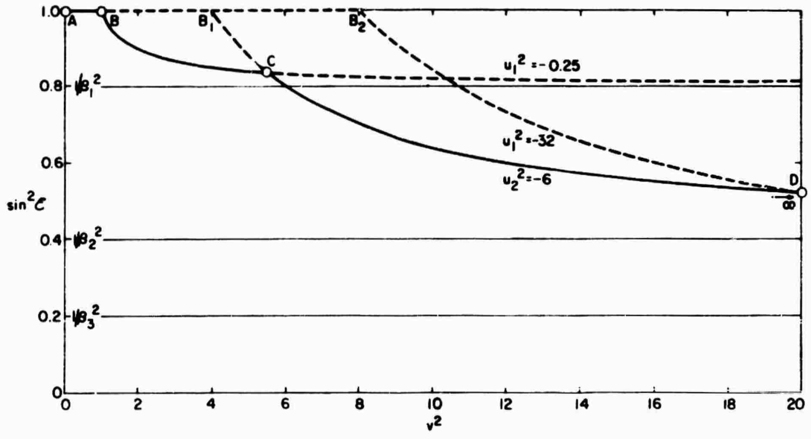

Figure 10. Limits of Integration for $0 > u_1^2 > u_2^2 > u_3^2$

From these examples it can be seen that three types of integrals are involved in the summation for I, namely,

$$I_\ell = \int_0^{v_{0i}} r^2 \frac{v^2}{2} E(v)\, dv,$$

$$I_m = \int_{v_{ij}}^{v_{jk}} r_j^2 \frac{v^2}{2} \left[1 - \frac{u_j^2}{v^2}\right] E(v)\, dv \quad ,$$

and

$$I_u = \int_{v_{jn}}^{\infty} r_n^2 \frac{v^2}{2} \left[1 - \frac{u_n^2}{v^2}\right] E(v)\, dv.$$

The summation is to be performed over the active grids taken in sequence from the outermost to the innermost grids. We will first evaluate these three integral types and then develop the summation for I for a general case having n grids.

For brevity in writing out the resulting equations, let

$$\exp(-ax^2) \equiv E(x)$$

and

$$\text{erf}(\sqrt{a}\,x) \equiv F(x)$$

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

by definition.

The above integrals can now be expressed in terms of

$$I_0 = \int_A^B E(v - v_0)\, dv - \int_A^B E(v + v_0)\, dv$$

and

$$I_2 = \int_A^B v^2\, E(v - v_0)\, dv - \int_A^B v^2\, E(v + v_0)\, dv$$

which are both standard forms.

$$\int_A^B E(v + v_0)\, dv = \frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}} \left[F(B + v_0) - F(A + v_0)\right]$$

$$\int_A^B v^2\, E(v + v_0)\, dv = \frac{1}{2a}\,(B - v_0)\, E(B + v_0) - \frac{1}{2a}\,(A - v_0)\, E(A + v_0) -$$

$$- \frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}} \left(v_0^2 + \frac{1}{2a}\right)\left[F(B + v_0) - F(A + v_0)\right] \quad .$$

Using these general expressions we may, upon inserting the appropriate limits A and B, evaluate the three desired integrals I_ℓ, I_m, and I_u, noting that for I_ℓ, when $A \rightarrow 0$

$$E(A + v_0) \rightarrow E(v_0) \quad ,$$

and

$$F(A + v_0) \rightarrow F(v_0) \quad .$$

For I_u when $B \rightarrow \infty$,

$$E(B + v_0) \rightarrow 0 \quad ,$$

and

$$F(B + v_0) \rightarrow 1 \quad .$$

$$I_\ell = \int_0^{v_{0i}} r^2 \frac{v^2}{2} E(v)\, dv = \frac{r^2}{2} \left[I_2 \right]_{A=0}^{B=v_{0i}}$$

$$I_\ell = \frac{r^2}{2} \left\{ \frac{v_0}{2a} \left[E(v_0) - E(v_{0i} + v_0) + E(v_0) + E(v_{0i} - v_0) \right] - \frac{v_{01}}{2a} \right.$$

$$\left[E(v_{0i} - v_0) - E(v_{0i} + v_0) \right] - \frac{\left(av_0^2 + \frac{1}{2} \right)}{2a} \left(\frac{\pi}{a} \right)^{\frac{1}{2}} \left[F(v_{0i} + v_0) - \right.$$

$$\left. \left. - F(v_{0i} - v_0) - F(v_0) - F(v_0) \right] \right\}$$

$$I_\ell = \frac{r^2}{2} \left\{ \left(av_0^2 + \frac{1}{2} \right) \left(\frac{\pi}{a} \right)^{\frac{1}{2}} F(v_0) + \frac{v_0}{a} E(v_0) - \frac{\left(av_0^2 + \frac{1}{2} \right)}{2a} \left(\frac{\pi}{a} \right)^{\frac{1}{2}} \left[F(v_{0i} + v_0) - F(v_{0i} - v_0) \right] - \right.$$

$$\left. - \frac{v_{0i} + v_0}{2a} E(v_{0i} - v_0) + \frac{v_{0i} - v_0}{2a} E(v_{0i} + v_0) \right\} \quad .$$

$$I_m = \int_{v_{ij}}^{v_{jk}} r_j^2 \frac{v^2}{2}\left[1 - \frac{u_j^2}{v^2}\right] E(v)\,dv = \frac{r_j^2}{2} \int_{v_{ij}}^{v_{jk}} v^2\, E(v)\,dv - \frac{r_j^2\, u_j^2}{2} \int_{v_{ij}}^{v_{jk}} E(v)\,dv.$$

$$I_m = \frac{r_j^2}{2}\left[I_2\right]_{A=v_{ij}}^{B=v_{jk}} - \frac{r_j^2\, u_j^2}{2}\left[I_0\right]_{A=v_{ij}}^{B=v_{jk}}$$

$$I_m = \frac{r_j^2}{2}\left\{ - \frac{\left(av_0^2 + \frac{1}{2}\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left[F({}_{jk} + v_0) - F({}_{jk} - v_0)\right] + \frac{\left(av_0^2 + \frac{1}{2}\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\right.$$

$$\left[F(v_{ij} + v_0) - F(v_{ij} - v_0)\right] - \frac{v_{jk} + v_0}{2a}\, E(v_{jk} - v_0) + \frac{v_{jk} - v_0}{2a}\, E(v_{ij} + v_0) +$$

$$\left. + \frac{v_{ij} + v_0}{2a}\, E(v_{ij} - v_0) - \frac{v_{ij} - v_0}{2a}\, E(v_{ij} + v_0)\right\} - \frac{r_j^2\, u_j^2}{2}\left\{ - \frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\right.$$

$$\left.\left[F(v_{jk} + v_0 - F(v_{jk} - v_0)\right] + \frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left[F(v_{ij} + v_0) - F(v_{ij} - v_0)\right]\right\}$$

$$I_m = \frac{r_j^2}{2}\left\{ - \left(\frac{av_0^2 + \frac{1}{2} - au_j^2}{2a}\right)\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left[F(v_{jk} + v_0) - F(v_{jk} - v_0)\right] - \right.$$

$$- \frac{v_{jk} + v_0}{2a}\, E(v_{jk} - v_0) + \frac{v_{jk} - v_0}{2a}\, E(v_{jk} + v_0) + \frac{\left(av_0^2 + \frac{1}{2} - au_j^2\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}$$

$$\left.\left[F(v_{ij} + v_0) - F(v_{ij} - v_0)\right] + \frac{v_{ij} + v_0}{2a}\, E(v_{ij} - v_0) - \frac{v_{ij} - v_0}{2a}\, E(v_{ij} + v_0)\right\}$$

$$I_u = \int_{v_{jn}}^{\infty} r_n^2 \frac{v^2}{2}\left[1 - \frac{u_n^2}{v^2}\right] E(v)\,dv = \frac{r_n^2}{2}\left[I_2\right]_{A=v_{jn}}^{B=\infty} - \frac{r_n^2}{2}\, u_n^2 \left[I_0\right]_{A=v_{jn}}^{B=\infty}$$

$$I_u = \frac{r_n^2}{2}\left\{\frac{v_0}{2a}\left[E(v_{jn}+v_0)+E(v_{jn}-v_0)\right]+\frac{v_{jn}}{2a}\left[E(v_{jn}-v_0)-E(v_{jn}+v_0)\right]-\right.$$

$$-\frac{1}{2}\frac{\left(av_0^2+\frac{1}{2}\right)}{a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left[-F(v_{jn}+v_0)+F(v_{jn}-v_0)\right]+\frac{u_n^2}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}$$

$$\left[-F(v_{jn}+v_0)+F(v_{jn}-v_0)\right]=\frac{r_n^2}{2}\left\{\frac{\left(av_0^2+\frac{1}{2}-au_n^2\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\right.$$

$$\left.\left[F(v_{jn}+v_0)-F(v_{jn}-v_0)\right]+\frac{v_{jn}+v_0}{2a}E(v_{jn}-v_0)-\frac{v_{jn}-v_0}{2a}E(v_{jn}-v_0)\right\}.$$

Writing out the resulting expression for Example 5, we note that the first integral is of the type I_m with $v_{ij} = u_1$, $v_{jk} = v_{12}$; the second has $v_{ij} = v_{12}$, $v_{jk} = v_{23}$; and the last is I_u with $v_{jn} = v_{23}$. The sequence of intercepts is u_1, v_{12}, v_{23}, ∞, giving for I:

$$I = P\left[\frac{r_1^2}{2}\left\{-\frac{\left(av_0^2+\frac{1}{2}-au_1^2\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left[F(v_{12}+v_0)-F(v_{12}-v_0)\right]-\frac{v_{12}+v_0}{2a}E(v_{12}-v_0)+\right.\right.$$

$$+\frac{v_{12}-v_0}{2a}E(v_{12}+v_0)+\frac{\left(av_0^2+\frac{1}{2}-au_1^2\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left[F(u_1+v_0)-F(u_1-v_0)\right]+$$

$$\left.+\frac{u_1+v_0}{2a}E(u_1-v_0)-\frac{u_1-v_0}{2a}E(u_1+v_0)\right\}+\frac{r_2^2}{2}\left\{-\frac{\left(av_0^2+\frac{1}{2}-au_2^2\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\right.$$

$$\left[F(v_{23}+v_0)-F(v_{23}-v_0)\right]-\frac{v_{23}+v_0}{2a}E(v_{23}-v_0)+\frac{v_{23}-v_0}{2a}E(v_{23}+v_0)+$$

$$+\frac{\left(av_0^2+\frac{1}{2}-au_2^2\right)}{2a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left[F(v_{12}+v_0)-F(v_{12}-v_0)\right]+\frac{v_{12}+v_0}{2a}E(v_{12}-v_0)-$$

$$- \frac{v_{12} - v_0}{2a} E(v_{12} + v_0)\Bigg\} + \frac{r_3^2}{2}\Bigg\{\left(\frac{av_0^2 + \frac{1}{2} - au_3^2}{2a}\right)\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\Big[F(v_{23} + v_0) - F(v_{23} - v_0)\Big] +$$

$$+ \frac{v_{23} + v_0}{2a} E(v_{23} - v_0) - \frac{v_{23} - v_0}{2a} E(v_{23} + v_0)\Bigg\}\Bigg] \; .$$

Collecting terms together,

$$I = \frac{P}{2}\Bigg[r_1^2\Bigg\{\left(\frac{av_0^2 + \frac{1}{2} - au_1^2}{2a}\right)\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\Big[F(u_1 + v_0) - F(u_1 - v_0)\Big] + \frac{u_1 + v_0}{2a} E(u_1 - v_0) -$$

$$- \frac{u_1 - v_0}{2a} E(u_1 + v_0)\Bigg\} + \left(r_2^2 - r_1^2\right)\Bigg\{\left(\frac{av_0^2 + \frac{1}{2} - av_{12}^2}{2a}\right)\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\Big[F(v_{12} + v_0) -$$

$$- F(v_{12} - v_0)\Big] + \frac{v_{12} + v_0}{2a} E(v_{12} - v_0) - \frac{v_{12} - v_0}{2a} E(v_{12} + v_0)\Bigg\} +$$

$$+ \left(r_3^2 - r_2^2\right)\Bigg\{\left(\frac{av_0^2 + \frac{1}{2} - av_{23}^2}{2a}\right)\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\Big[F(v_{23} + v_0) - F(v_{23} - v_0)\Big] +$$

$$+ \frac{v_{23} + v_0}{2a} E(v_{23} - v_0) - \frac{v_{23} - v_0}{2a} E(v_{23} + v_0)\Bigg\}\Bigg] .$$

For Case 6, the sequence of intercepts is 0, v_{01}, v_{12}, v_{23}, ∞, giving, in the same way as 5

$$I = \frac{P}{2}\Bigg[r^2\Bigg\{\left(\frac{av_0^2 + \frac{1}{2}}{2a}\right)\left(\frac{\pi}{a}\right)^{\frac{1}{2}} 2F(v_0) + \frac{v_0}{2a} 2E(v_0)\Bigg\} + (r_1^2 - r^2)\Bigg\{\left(\frac{av_0^2 + \frac{1}{2} - av_{01}^2}{2a}\right)$$

$$\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\Big[F(v_{01} + v_0) - F(v_{01} - v_0)\Big] + \frac{v_{01} + v_0}{2a} E(v_{01} - v_0) - \frac{v_{01} - v_0}{2a}$$

$$E(v_{01} + v_0)\Bigg\} + (r_2^2 - r_1^2)\ \{\ldots\}\ + (r_3^2 - r_2^2)\ \{\ldots\}\ \Bigg] \; .$$

Thus for a sequence of active grids, r_i, $i = 0, 1, 2, 3, \ldots n$.

$$I = \frac{P}{2} \sum_{i=0,1}^{n} \left[\left(r_i^2 - r_{i-1}^2 \right) \left\{ \frac{\left(av_0^2 + \frac{1}{2} - av_{i-1,i}^2 \right)}{2a} \left(\frac{\pi}{a} \right)^{\frac{1}{2}} \left[F(v_{i-1,i} + v_0) - F(v_{i-1,i} - v_0) \right] + \right.\right.$$

$$\left.\left. + \frac{v_{i-1,i} + v_0}{2a} E(v_{i-1,i} - v_0) - \frac{v_{i-1,i} - v_0}{2a} E(v_{i-1,i} + v_0) \right\} \right] .$$

Putting

$$\gamma^2 = av_0^2 = \frac{m}{2kt} v_0^2 ,$$

$$x_{(i-1),i}^2 = av_{(i-1),i}^2 = \frac{r_{i-1}^2 au_{i-1}^2 - r_i^2 au_i^2}{r_{i-1}^2 - r_i^2} ,$$

where $au_i^2 = \frac{eV_i}{kT}$; we then get

$$F(v_{(i-1),i} \pm v_0) = \{ \mathrm{erf} \sqrt{a}\, v_{(i-1),i} \pm \sqrt{a}\, v_0 \} = \mathrm{erf} \{ x_{i-1,1} \pm \gamma \} .$$

and

$$E(v_{(i-1),i} \pm v_0) = \exp \left\{ -(\sqrt{a} v_{(i-1),i} \pm \sqrt{a}\, v_0)^2 \right\} = \exp \left\{ - (x_{(i-1),i} \pm \gamma)^2 \right\} .$$

Taking $\frac{1}{a\sqrt{a}} = \left(\frac{2kT}{m} \right)^{\frac{3}{2}}$ outside the summation, we have

$$\frac{P}{2a\sqrt{a}} = \frac{\pi Ne}{\sqrt{\pi a}} \frac{1}{\gamma} = \frac{\pi Ne}{2} \sqrt{\frac{8kT}{\pi m}} \frac{1}{\gamma}$$

where $P = 2\pi Ne\left(\frac{a}{\pi}\right)^{\frac{1}{2}}\frac{1}{v_0}$.

Hence

$$I = \frac{\pi Ne}{2}\sqrt{\frac{8kT}{\pi m}}\ \frac{1}{\gamma}\sum_i (r_i^2 - r_{i-1}^2)\left\{\left(\frac{1}{2} + \gamma^2 - x_{(i-1),i}^2\right)\frac{\sqrt{\pi}}{2}\right.$$

$$\left[\operatorname{erf}\left\{x_{(i-1),i} + \gamma\right\} - \operatorname{erf}\left\{x_{(i-1),i} - \gamma\right\}\right] + \frac{x_{(i-1),i} + \gamma}{2}$$

$$\left.\exp\left\{-(x_{(i-1),i} - \gamma)^2\right\} - \frac{x_{(i-1),i} - \gamma}{2}\exp\left\{-(x_{(i-1),i} + \gamma)^2\right\}\right\} .$$

Where, if the first active grid is retarding , that is, $eV_1 \geq 0$, $u_1^2 \geq 0$, summation starts at $i = 1$ with $r_{i-1} = r_0 = 0$. For the first active grid accelerating, $eV_1 < 0$, $u_1^2 < 0$, summation starts at $i = 0$ with $r_{i-1} = r_{-1} = 0$. Note that with $V_0 = 0$ (by definition) and $r_0 = r$ for the retarding first grid:

$$(r_1^2 - r_0^2) = r_1^2 \qquad \text{and} \qquad x_{01}^2 = \frac{r_0^2\, au_0^2 - r_1^2\, au_1^2}{r_0^2 - r_1^2} = au_1^2 \quad ;$$

and for an accelerating first grid:

$$(r_0^2 - r_{-1}^2) = r_0^2 = r^2 \quad , \quad x_{-10}^2 = \frac{r_{-1}^2\, au_{-1}^2 - r_0^2\, au_0^2}{r_{-1}^2 - r_0^2} = 0$$

and

$$x_{01}^2 = \frac{r_0^2\, au_0^2 - r_1^2\, au_1^2}{r_0^2 - r_1^2} = \frac{r_1^2\, au_1^2}{r^2 - r_1^2} \quad .$$

The method used to determine the active grid sequence i = 0 or 1 through n from the total grid sequence K = 0 through N may be summarized as follows:

(a) Compare the sequence of intercepts

$$x_{0k}^2 = -\frac{r_k^2\, au_k^2}{r^2 - r_k^2} \quad ,$$

k = 1 through N, and find the grid K = k_1 having the least x_{0k}^2, this $k_1{}^{th}$ grid is the first active grid. If this value x_{0k}^2 is negative or zero, this k^{th} grid is retarding and i = 1. If two or more grids give the same intercept, the one with the higher k is chosen.

(b) Compare the sequence of intercepts

$$x_{k_1 k} = \frac{r_{k_1}^2\, au_{k_1}^2 - r_k^2\, au_k^2}{r_{k_1}^2 - r_k^2} \quad ,$$

k = k_1 + 1 through N, again finding the grid k = k_2 having the least $x_{k_1 k}^2$, this grid is now the second active grid (i = 2).

(c) Repeat step (b) until k_n = N when the active grid sequence is complete (i = n). Note that the N^{th} (innermost) grid is always an active grid because its limiting curve, whether accelerating or retarding, tends asymptotically to $\sin^2 \epsilon = 1/\beta_N^2 = r_N^2/r^2$ as $v^2 \rightarrow \infty$, thus having the least value of $\sin^2\epsilon$ of all the grids in the system and being the final integral in the sum for I.

From the foregoing discussion of a system of concentric spherical grids placed in a drifting Maxwellian plasma, we conclude that the current flow through the innermost grid:

(a) is a function of the plasma parameters: charged particle number density N, particle temperature T, particle mass m, particle charge e, and the drift speed v_0;

(b) is not necessarily dependent upon all grid radii and the potentials thereon;

(c) if one or more grids are retarding, is independent of any grids of radii greater than the most retarding grid, including the outermost. If, however, no grids are retarding, the current flow is dependent on the outer grid radius but independent of any grids of radii greater than that of the "least accelerating" grid.

11. APPLICATIONS

The relations derived in the preceeding sections can be used to obtain the plasma parameters from the observed current-voltage curve for the probe.

Since the various expressions all involve particles of only one sign - it is of interest, at this point, to investigate the possibility of using a probe which responds to particles of one sign only, rejecting the other. To this end, it seems that a reasonable method would be to put a large enough bias on the collecting surface to repel all particles of one sign, but to surround this element with a perforated grid so that the effects of this bias will not penetrate the surrounding plasma. This, then, gives us a two-element sensor. The final conclusions in Section 10 stated that the inward-current flow is a summation over all active grids and always includes a term containing the potential and radius of the innermost grid, in the present case, the collector. This term is for an attractive potential V_2

$$T_2 = (r_1^2 - r_2^2)\left\{\left(\frac{1}{2} + \gamma^2 - x_{12}^2\right)\frac{\sqrt{\pi}}{2}\left[\operatorname{erf}\{x_{12} + \gamma\} - \operatorname{erf}\{x_{12} - \gamma\}\right] + \right.$$
$$\left. + \frac{x_{12} + \gamma}{2}\exp\{-(x_{12} - \gamma)^2\} - \frac{x_{12} - \gamma}{2}\exp\{-(x_{12} + \gamma)^2\}\right\}$$

It can be shown that if $x_{12}^2 \gg \gamma^2$ and $x_{12}^2 \gg 1$ the above term tends to zero, and the current flow will be independent of the inner grid potential for the attracted sign particle. For the retarded particle, however, the total current flow has only one term (under the assumption that $|V_2| \gg |V_1|$), namely,

$$I = \frac{\pi Ne}{2}\sqrt{\frac{8kt}{\pi m}}\,\frac{1}{\gamma}\,r_2^2\left\{\left(\frac{1}{2} + \gamma^2 - x_{02}^2\right)\frac{\sqrt{\pi}}{2}\left[\operatorname{erf}\{x_{02} + \gamma\} - \operatorname{erf}\{x_{02} - \gamma\}\right] + \right.$$
$$\left. + \frac{x_{02} + \gamma}{2}\exp\{-(x_{02} - \gamma)^2\} - \frac{x_{02} - \gamma}{2}\exp\{-(x_{02} + \gamma)^2\}\right\} .$$

Again, if $x_{02}^2 \gg \gamma^2$ and $x_{02}^2 \gg 1$, then the total current will tend to zero and there will be no contribution from the retarded sign particle.

Putting

$$x_{12}^2 = \frac{r_1^2 a u_1^2 - r_2^2 a u_2^2}{r_1^2 - r_2^2} \quad , \qquad a u_n^2 = \frac{eV_n}{kT} \quad , \qquad a = \frac{m}{2kT} \quad , \quad \text{and} \quad \gamma = a v_0^2 \quad ,$$

the conditions for attracted particle 'saturation' become:

(a) for $\gamma \geq 1$, $x_{12}^2 \gg \gamma^2$,

$$e(-V_2) \gg \frac{1}{2} m v_0^2 \left(\frac{r_1^2}{r_2^2} - 1 \right) - \frac{r_1^2}{r_2^2} eV_1 \quad ;$$

and

(b) for $\gamma < 1$, $x_{12}^2 \gg 1$,

$$e(-V_2) \gg kT \left(\frac{r_1^2}{r_2^2} - 1 \right) - \frac{r_1^2}{r_2^2} eV_1 \quad .$$

On the other hand, for the retarded particle, putting

$$x_{02}^2 = - \frac{r_2^2 a u_2^2}{-r_2^2} = a u_2^2 = \frac{eV_2}{kT} \quad ,$$

(a) for $\gamma \geq 1$, $x_{02}^2 \gg \gamma$,

$$e(V_2) \gg \frac{1}{2} m v_0^2 \quad ;$$

and

(b) for $\gamma < 1$, $x_{02}^2 \gg 1$,

$$e\,(V_2) \gg kT.$$

Since the equations for the attracted particles involve the geometry of the sensor, and those for the retarded particles do not, and since one potential must satisfy both sets of equations; the sensor design, as well as the expected temperatures and velocities, will determine which set of equations imposes the most stringent limit on V_2.

On the assumption that the bias potential on the collecting electrode satisfies the set of conditions, we may proceed with the analysis on the basis of having a spherical probe of radius $R(=r_1)$ at a potential $V(=V_1)$ measuring particles of one sign only.

Not all of the particles which are incident upon the sphere with radius equal to that of the outer (perforated) grid will contribute to the current since some will strike the grid wires. The observed current is reduced by the transmission factor of the outer grid ξ. The true current I will, therefore, be the measured current divided by ξ. A further complication arises from the fact that all potentials in the preceding sections are referred to the plasma potential which is unknown. In practice, the sweep potential applied to the probe must be referred to some other point in the system such as one of the electrodes in a discharge tube or the skin of a rocket in a rocket experiment. The plasma potential is one of the unknowns we would like to determine. In the following, V is understood to be measured from the reference potential, and it is assumed that the potential difference between the reference electrode and the plasma remains constant regardless of the potential applied to the probe.

First, we consider the case for no-drift. In the retarding region, for the relation between I and V, we can write

$$I = I_B \exp\left(-\frac{eV}{kT}\right) \quad .$$

To simplify the discussion we will consider the case of electrons, and defining $\alpha = |e|/kT$, we have

$$I = I_B \exp(\alpha V) \quad .$$

In the "near accelerating" region, where the current may be represented by the first two terms of the Taylor expansion, we can write

$$I = a(V - V_A) \quad .$$

We have now introduced four constants: I_B, α, a, and V_A. They are not, however, all independent, since we have shown that the gradient (dI/dV) must be the same at the plasma potential for both retarding and accelerating expressions. Differentiating the above expressions gives

$$\frac{dI}{dV} = \alpha I \qquad \text{retarding}$$

$\frac{dI}{dV} = a$ accelerating

At the plasma potential V_0 the current flow will be I_0. Equating these two gradients at the plasma potential gives

$$\alpha I_0 = a,$$

or

$$I_0 = \frac{a}{\alpha} \quad .$$

This gives us a means of obtaining the current at plasma potential I_0. We plot log I versus V, which will be a straight line in the retarding region. The slope of this line is α. A plot of I versus V will give a straight line in the near accelerating region. If the conditions for the particular experiment are appropriate, there will be a reasonably large range of voltage over which the two-term Taylor expression will be a good approximation, and this straight line will be readily located. Its slope is a and the current at plasma potential is obtained as the ratio of the two slopes. Note that the determination of this current did not involve an accurate determination of the plasma potential. The slope α is simply related to the temperature, hence, electron temperature is obtained from the equation

$$T = \frac{e}{k} \frac{1}{\alpha} \quad .$$

The electron density is given by

$$I_0 = 2.48821 \times 10^{-16} \, (4\pi R^2) \, N \sqrt{T} \quad .$$

The intercept of the straight line V_A can also be determined, and from the equation

$$I_0 = a(V_0 - V_A) \quad ,$$

we have

$$V_0 = \frac{1}{\alpha} + V_A \quad ,$$

which gives the plasma potential.

We have now determined three plasma parameters from the current-voltage characteristic of the probe, namely, the electron temperature, the electron density, and the plasma potential. This is the maximum number that can be obtained. The curve fitting process introduces four parameters as already noted. However, there is one relationship between them, that is, the condition to match the slopes of the two curves at plasma potential. Thus, there are only three independent parameters.

This particular point has more significance for positive ions when the mass m is also an unknown. In this case only three of the four unknowns can be obtained from the probe characteristic; the temperature is obtained from α, the plasma potential from α and V_A, and then using a in conjunction with α to give I_0, we get $N/\sqrt{m}$. Hence, either m must be known so that N can be calculated, or vice versa.

When the drift velocity cannot be neglected, the equations are much more involved. The straight line part of the accelerating characteristic now extends into the retarding region as noted earlier but even with large drift velocities, if x (that is, voltage) is made large enough, a curvature in the characteristic will eventually occur. This is likely to be of limited practical importance, since making the voltage large for the retarding case implies very small currents which will probably be less than the noise level. Even if this can be observed, there is still no simple analytic procedure which can be applied to it comparable to that which we can use in the case for no-drift. However, there is still the straight line part of the I-V characteristic (which now extends over parts of both the retarding and accelerating regions) from which a slope and an intercept may be obtained.

The slope obtained from this line is given by

$$\frac{2\pi R^2 Ne^2}{mv_0} \operatorname{erf}\left(v_0 \sqrt{\frac{m}{2kT}}\right).$$

This expression contains the density N, the mass m, and the temperature T, all of which are unknowns. Note that the plasma potential does not enter into this expression. To obtain an expression for the intercept, suppose we fit the expression

$$I = a(V - V_A)$$

to the data so that V_A is the intercept on the voltage axis. For the plasma potential we have

$$I_0 = a(V_0 - V_A) \quad ,$$

and since a is the slope,

$$I_0 = \left(\frac{dI}{dV}\right)_0 (V_0 - V_A) \quad .$$

Using the expressions already developed for I_0 and $(dI/dV)_0$ we obtain

$$V_0 - V_A = \mp \frac{mv_0^2}{2e}\left[\left(\frac{1}{2\gamma^2} + 1\right) + \frac{1}{\gamma\sqrt{\pi}} \; \frac{\exp(-\gamma^2)}{\operatorname{erf}(\gamma)}\right] \quad ,$$

where the upper sign applies to positive particles and the lower to negative particles. This expression involves the three unknowns: plasma potential V_0, the mass m, and the temperature T, which occur in γ. Note that the density N does not enter into this expression.

We now have two relations between four unknowns. These enter into the expressions in such a way, that if independent estimates can be made of <u>any</u> two unknowns, the other two can be determined. However, some cases will be much simpler to compute than others. For example, solving for V_0 and N, when m and T are known, is straightforward. On the other hand, solving for m requires a numerical iteration process.

As an example of how the procedure might apply in practice, consider the case of a rocket carrying two probes —one measuring electrons, and the other, positive ions. In practice, due to the high thermal velocity of the electrons, the rocket velocity can be neglected for the electron probe, however, for the ion probe, γ is likely to have a value of about unity. For the electron probe there will be only three unknowns since the electron mass is known. From the retarding part of the characteristic we obtain electron temperature, and using the accelerating characteristic as well, gives density and plasma potential. We might now assume electrical neutrality so that the ion density is equal to the electron density. Since the plasma potential has already been determined, we have V_0 and N for the positive probe and can use the probe results to obtain mass and temperature. Alternatively, we might assume that the ion temperature is equal to the neutral temperature which is known from independent experiments and determine N and m from the probe data, or we might use independent results on composition to obtain m and determine N and T.

Appendix

We list below the equations we have derived, together with some of their special forms. The following notations are used throughout:

I = current flowing to the spherical collector,
V = potential difference between collector and plasma,
R = radius of collector,
r = radius of sheath,
N = charged particle density in the plasma,
e = charge on the particle,
m = mass of charged particle,
k = Boltzmann's constant,
T = Temperature of charged particle being collected,
v_0 = drift velocity of plasma relative to collector,
$a = m/2kT$,
$\gamma = \sqrt{a}\, v_0$
$\alpha = R^2/(r^2 - R^2)$,

$$x = \begin{cases} |eV/kT & \text{for retarding voltages,} \\ (R^2/(r^2 - R^2)(|eV|/kT) & \text{for accelerating voltages,} \end{cases}$$

$$\operatorname{erf} y = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-t^2)dt.$$

(a) Retarding Expression:-

$$I = \pi R^2 Ne \frac{1}{\sqrt{\pi}a} \frac{1}{\gamma} \left[\frac{\pi}{2} \left(\frac{1}{2} + \gamma^2 - x^2 \right) \left\{ \mathrm{erf}\,(x + \gamma) - \mathrm{erf}\,(x - \gamma) \right\} + \right.$$

$$\left. + \frac{1}{2}(x + \gamma) \exp \left\{ -(x - \gamma)^2 \right\} - \frac{1}{2}(x - \gamma) \exp \left\{ -(x + \gamma)^2 \right\} \right].$$

(b) Accelerating Expression:-

$$I = \pi r^2 \; Ne \frac{1}{\sqrt{\pi}a} \frac{1}{\gamma} \left[\frac{\sqrt{\pi}}{2} (1 + 2\gamma^2)\, \mathrm{erf}\, \gamma + \gamma \exp(-\gamma^2) - \frac{1}{\alpha + 1} \left\{ \frac{\pi}{2} \left(\frac{1}{2} + \gamma^2 - x^2 \right) \right. \right.$$

$$\left[\mathrm{erf}\,(x + \gamma) - \mathrm{erf}\,(x - \gamma) \right] + \frac{1}{2}(x + \gamma) \exp \left[-(x - \gamma)^2 \right] -$$

$$\left. \left. - \frac{1}{2}(x - \gamma) \exp \left[-(x + \gamma)^2 \right] \right\} \right] .$$

(c) Taylor Expansion of Accelerating Expression:-

$$I = \pi R^2 \; Ne \; v_0 \left[\left(1 + \frac{1}{2\gamma^2} \right) \mathrm{erf}\,(\gamma) + \frac{2}{\sqrt{\pi}} \frac{1}{2\gamma} \exp(-\gamma^2) \mp \frac{1}{\gamma^2} \left| \mathrm{erf}\,(\gamma) \right| \left(\frac{eV}{kT} \right) - \right.$$

$$\left. - \frac{1}{2} \frac{2}{\sqrt{\pi}} \frac{1}{2\gamma} \left| \exp(-\gamma^2) \right| \left(\frac{eV}{kT} \right)^2 + \ldots \right] ,$$

where the upper sign is for positive particles, and the lower for negative.

(d) Current at Plasma Potential (V = 0):-

$$I_0 = \pi R^2 \; Ne \; v_0 \left[\left(1 + \frac{1}{2\gamma^2} \right) \mathrm{erf}\, \gamma + \frac{2}{\sqrt{\pi}} \frac{1}{\gamma} \exp(-\gamma^2) \right] .$$

(e) Derivative of the Current at Plasma Potential:-

$$\left(\frac{dI}{dV} \right)_0 = \mp \frac{2\pi R^2 \; Ne^2}{mv_0} \; \mathrm{erf}\, \gamma .$$

where the upper sign is for positive particles, and the lower for negative.

When the drift velocity is zero, the above expressions reduce as follows:

(f) Retarding Expression:-

$$I = 4\pi R^2 \, Ne \sqrt{\frac{kT}{2\pi m}} \exp\left(-\frac{eV}{kT}\right) \quad .$$

(g) Accelerating Expression:-

$$I = 4\pi r^2 \, Ne \sqrt{\frac{kT}{2\pi m}} \left[1 - \frac{r^2 - R^2}{r^2} \exp\left(-\frac{R^2}{r^2 - R^2} \frac{eV}{kT}\right)\right] \quad .$$

(h) Taylor Expansion of Accelerating Expression:-

$$I = 4\pi R^2 \, Ne \sqrt{\frac{kT}{2\pi m}} \left[1 \mp \frac{eV}{kT} - \frac{1}{4} \frac{R^2}{r^2 - R^2} \left(\frac{eV}{kT}\right)^2 \cdots \right] \quad .$$

(i) Current at Plasma Potential:-

$$I_0 = 4\pi R^2 \, Ne \sqrt{\frac{kT}{2\pi m}} \quad .$$

Note that the average particle velocity in a Maxwell distribution is given by

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}} \quad ,$$

so that the above current can be expressed as

$$I_0 = \pi R^2 \, Ne \, \bar{v} \quad .$$

This checks with the result obtained from elementary kinetic theory— the number of particles crossing unit area in unit time is given by $N\bar{v}/4$.

The expression for I_0 contains the universal constants e and k and the numerical values may be inserted. Taking $e = 1.60210 \times 10^{-19}$ coulomb, $k = 1.38054 \times 10^{-23}$ joule/°K and considering electrons for which $m = 9.1091 \times 10^{-31}$ kg, we obtain

$$I_0 = 2.48821 \times 10^{-16} (4\pi R^2) N \sqrt{T},$$

where I_0 is the current in amperes, R is the distance in meters, and N equals the particles per cubic meter.

Considering positive ions of atomic mass M, and taking the atomic mass unit to be 1.65979×10^{-27} kg, we have

$$I_0 = 5.82906 \times 10^{-18} (4\pi R^2) N \sqrt{\frac{T}{M}} .$$

(j) Derivative of Current at Plasma Potential:-

$$\left(\frac{dI}{dV}\right)_0 = \mp 4\pi R^2 Ne^2 \sqrt{\frac{1}{2\pi mkT}} .$$

Substituting numerical values as above, we have for electrons

$$\left(\frac{dI}{dV}\right)_0 = 2.88753 \times 10^{-12} (4\pi R^2) \frac{N}{\sqrt{T}} ,$$

and for positive ions,

$$\left(\frac{dI}{dV}\right)_0 = 6.76453 \times 10^{-14} (4\pi R^2) \frac{N}{\sqrt{TM}}$$

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Michael Smiddy
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13. ABSTRACT

A theoretical derivation of the characteristic of a spherical probe in a flowing Maxwellian plasma is given together with various limiting forms of the equations. The limitations imposed on the validity of the theory by the characteristics of the sheath are discussed. The theory is then extended to the case of an n-grid sensor. Finally, the application of the theory to the determination of plasma parameters from the probe characteristic is discussed.

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